Non-cooperative Equilibria as Walrasian Models of Exchange

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Abstract

The role of Walrasian prices in exchange can be systematically applied as a rationale for non-cooperative equilibria in games. Similarities are highlighted by adopting the polyhedral convexity in (finite) normal form games as a setting for exchange. Probabilities and strategies in games are regarded, respectively, as prices and activities found in exchange. Commodity excess demands derived from price-taking maximization in exchange have their counterparts in vectors of deviation gain-vectors derived from price-taking maximization in games.

A distinction between complete duality and incomplete duality characterizes the difference, respectively, between (i) quasi-linear utility and (ii) non-quasi-linear utilities in exchange. The same duality distinction characterizes the difference between (iii) Hannan and (iv) Correlated equilibrium compared to (v) Nash equilibrium in games. Each of these equilibrium concepts shares a minimax property as the description of equilibrium.

Adjustment of prices to aggregate excess demands in exchange has its counterpart in adjustment of prices to aggregate deviation gains in games. In both exchange and games, the success/failure of convergence is tied to the complete/incomplete duality distinction.

Keywords:

Economics: Quasi-linear and non-quasi-linear Walrasian equilibrium, price-adjustment to excess demands.

Games: Nash, Hannan, and Correlated equilibrium, price-adjustment to excess demands.

Convex Analysis: Conjugate duality, minimax, subgradient algorithms.

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1 Introduction

Non-cooperative equilibrium is regarded as providing a strategic foundation missing in the Walrasian model of economic interdependence. Without disputing the importance of strategic behavior, the purpose of this paper is to demonstrate that the role of Walrasian prices in exchange can be systematically applied as a rationale for non-cooperative equilibria in games. The distinction is between price-taking maximization without interdependent utilities in exchange and price-taking maximization with interdependent utilities in games.

Gale, Kuhn, Tucker [1951] demonstrated the similarity between strategic and resource constraints by showing that the Minimax Theorem for two-person zero-sum games, the starting point for non-cooperative equilibrium, can be formulated as a linear programming problem, a canonical statement of scarcity pricing. These results were influential in the development of convex analysis, extending the minimax principle to a wider class of optimization problems underlying the origins of resource/strategic pricing.

A statement by Debreu [1952] unites general equilibrium with game theory:

> The existence theorem presented here gives general conditions under which there is for such a social system an equilibrium, i.e., a situation where *the action of every agent belongs to his restricting subset and no agent has incentive to choose another action*. This theorem has been used by Arrow and Debreu to prove the existence of an equilibrium for a classical competitive economic system, it contains the existence of an equilibrium point for an N-person game (Nash [1950]) and, naturally, as a still more particular case, the existence of a solution for a zero-sum two-person game (von Neumann and Morgenstern [1947]). ....In Section 3 saddle points are presented as particular cases of equilibrium points and in connection with the closely related MinMax operator. (Italics added.)

The presentation, below, elaborates on this theme.

Extensions of non-cooperative equilibrium by Hannan [1957] and Aumann [1974] extend the duality principles underlying two-person zero-sum games. And the study of general equilibrium models with quasi-linear utilities, stimulated by their connection to cooperative games with transferable utility, allows the duality underlying linear programming to be more directly linked to Walrasian equilibrium (WE). The unification theme will include quasi-linear (QL) and non-quasi-linear (non-QL) WE and their connections with Nash equilibrium (NE), Hannan equilibrium (HE), and Aumann's correlated equilibrium (CE).

In line with the focus, below, exchange terminology will define concepts in games. For example, probabilities are prices and strategies are referred to as activities.

A first step in the translation addresses the discrepancy between the Cournot-Nash description of individuals responding to *the choices of others* and the Walrasian depiction of individuals responding to *prices.* To illustrate how the former can be transformed into the latter, consider a 2-person problem where each has three activities, $A_i = \{a_{i1}, a_{i2}, a_{i3}\}, i = 1, 2$. Utilities for each i are defined for the 9 possible outcomes $a = (a_{1k}, a_{2h}) \in A, k, h = 1, 2, 3$. In Nash's formulation, there are 3 activities for each individual (and their respective convex combinations). Choices are depicted in the Cartesian product $(q_1, q_2) \in \Delta(A_1) \times \Delta(A_2) \in \mathbb{R}^6$, where $\Delta(A_i) \in \mathbb{R}^3$ is the unit simplex on A_i, allowing q_i to maximize directly against q_j, $i \neq j$. An equivalent formulation is to define the *tensor* product,

$$\mathfrak{T}(q_1, q_2) := q_1 \otimes q_2 \in \Delta(A_1) \otimes \Delta(A_2) \in \mathbb{R}^9$$

which has the same dimension as the space on which utilities are defined. From this perspective, both individuals react to the *same* $q_1 \otimes q_2$: i.e., mutual best response is translated to price-taking maximization. A two-person game will be regarded as an exchange economy with two individuals and a third entity that sets prices. The two players individually maximize with respect to $\mathfrak{T}(q_1, q_2)$ and the third wants to minimize their gains.

The tensor product formulation exhibits dimensional consistency with non-cooperative equilibria allowing joint prices/probabilities on A, i.e.,

$$\mathfrak{T}[Q] \subset P = \Delta(A) = \{p : p(a) \geq 0, \sum_{a \in A} p(a) = 1\} \subset \mathbb{R}^9.$$

Non-cooperative equilibria include two methods of pricing, while the standard model of exchange has one. NE is the restriction of prices to $\mathfrak{T}[Q]$, while HE allows P.

In addition, non-cooperative equilibria include the possibility of two different sets of price-takers, to be called *non-delegated* or *delegated*. Delegation expands the number of price-takers from from the (non-delegated) individuals to allow an agent of each individual to be in charge of choices for each activity $a_i \in A_i$. In the above example, there are two non-delegated choosers and 6 delegated ones. The opportunity to delegate is based on prices. For example, if $\sum_{a_2 \in A_2} p(a_1, a_2) = 0$, a_1's choice is irrelevant. Delegation as a distinctive feature of CE appeared in Ostroy and Song [2009].

The variety of price-taking combinations corresponds to different versions of non-cooperative equilibria illustrated in the following table, where **N** and **D** stand for non-delegated and delegated. Other than the bottom-right entry, each combination corresponds to a different

	P	$\mathfrak{T}$[**Q**]
N	HE	NE
D	CE	NE

definition of non-cooperative equilibrium. That the entries in the right column are the same expresses the conclusion that there are no gains to delegation when prices are restricted.

A formal discrepancy between games and exchange is that the former has a finite number of activities while standard presentations of the latter do not. As an accommodation to games to make comparisons more explicit, exchange will be modeled under the restriction of polyhedral convexity, consistent with a finite number of activities.

The relevance of activity choices (and their convex combinations) in exchange is determined by their associated *commodity* excess demand vectors: activities are merely stand-ins for commodities in exchange. While there are no commodities in a game, activities can be associated with a vector of *deviation gain* "excess demands." The duality between price and commodity excess demands in exchange is translated to price and deviation-gain vectors in games.

Equilibria are characterized by a minimax/saddle-point condition in both games and exchange. The different methods required to demonstrate this condition will be an organizing principle, below. These differences occur within, rather than between, games and exchange. They are attributable to the difference between what will be called a "complete duality" to establish equilibrium compared to an "incomplete duality." In words, with completeness — another name for conjugate duality — every possibly relevant bounding hyperplane to a convex set is allowed. Incompleteness means some are precluded. Completeness allows more elementary methods to demonstrate existence.

The QL model of exchange allows a complete duality. The added budgetary restriction imposed in the non-QL version of exchange is well-known to create "income effects of price changes." This is the source of incompleteness in exchange. The use of P allows a complete duality in games. The restriction to $\mathfrak{T}[Q]$ makes it incomplete.

A remarkable special class of *identical interest games* introduced by Monderer and Shapley [1996a,1996b] allows $\mathfrak{T}[Q]$ to provide a *complete-in-itself* duality for NE, where every relevant bounding hyperplane (and some extreme points) of the polyhedral convex set defining such a game are in $\mathfrak{T}[Q]$. The "complete-in-itself" duality occurs within the larger complete duality, admitting non-cooperative equilibria with prices in $P \backslash \mathfrak{T}[Q]$. Identical interest games represent an intermediate position between the problems caused by incompleteness for general games and two-person zero-sum games. When a two-person game is zero-sum, opportunities available to the price setter at P are the same as $\mathfrak{T}[Q]$.

The completeness/incompleteness distinction separating methods for *proving* existence has the property that *at equilibrium* the two methods of pricing coincide. The difference is only observed in maximizing responses away from equilibrium. Hence, differences in proving existence extend to the success/failure of methods for showing convergence to equilibrium. In exchange, the well-known failure of convergence to WE via prices adjusting to excess demands in the non-QL model will compared to its success in the polyhedral QL version of exchange in Section 6. Similar conclusions with respect to completeness/incompleteness conditions apply to excess demand versions of price adjustment for games. of games. In later work (Ostroy and Song [2023]), method extended to models of price adjustment applying specifically to games.

Section 2 contains a overview of polyhedral convexity. For games this is the duality between indicator functions of polyhedral sets and their support functions. For exchange, it is based on real-valued concave functions and their conjugates, defined on a polyhedral sets. Distinctions between completeness and incompleteness are based on the presence of absence of conjugate duality.

Section 3 is devoted to individual price-taking maximization. For exchange, this is divided into QL and non-QL versions in Section 3.1. For games, Section 3.2 defines maximization as it varies with respect to the two systems of prices and with respect to delegation versus non-delegation. Section 3.3 compares the different forms of maximizing behavior in 3.2.

The purpose of Section 4 is highlight essential similarities between equilibrium in exchange and games as sharing a minimax/saddle-point property, including how the completeness/incompleteness distinction determines the method of proving that property. Most existence results in Section 4 have been previously established, although not necessarily using the same characterization.

In exchange, WE with QL allows a complete duality underlying linear progamming; hence, a non-fixed-point proof of the minimax condition. With non-QL and incomplete duality, a fixed-point argument is well-known to be required for WE. This feature and its incomplete origin is shared by NE in games. For games, Hart and Schmeidler [1989], Nau and McCardle [1990], and Myerson [1997] show that duality methods suffice for existence of CE. HE has been regarded as an extension of CE amenable to similar treatment. Here it is pointed out that when HE is characterized by the minimax condition, the price setter may impose losses on the maximizing participants. (See Section 3.3.3) A modification in the definition of game is given to establish 0 as the lower bound for HE.

Section 5 is devoted to games of *identical interest* introduced by Monderer and Shapley [1996a,1996b] and two-person zero-sum (2-0) games having the common feature of using only prices $\mathfrak{T}[Q]$ without appeal to a fixed-point argument. The (deviation gain) polyhedron defined by an identical interest game contains a sub-polyhedron for which $\mathfrak{T}[Q]$ is effectively a complete duality. 2-0 games have the property that dualities defined by P and $\mathfrak{T}[Q]$ are equivalent.

Section 6 shows that the price adjustment model for exchange can be translated into price adjustment for games. The standard version for exchange uses continuous time and continuous (single-valued) excess demands. (A discrete time model of price adjustment for exchange with continuous demands is given by Uzawa [1960].) Discrete time models required for polyhedral convexity necessarily contain multi-valued and discontinuous excess demands. Nevertheless, with appropriate qualifications, Section 6.1 shows that convergence via subgradient algorithms can be established for WE with QL. Well-known non-convergence for non-QL WE (Scarf [1960]) applies to its polyhedral counterpart.

Section 6.2 shows that price adjustment subgradient algorithms for games converge for HE and CE. Because these algorithms converge to a equilibrium prices, they represent a constructive method of demonstrating existence of HE and CE. The non-convexity of $\mathfrak{T}[Q]$ as a subset of P precludes application of a subgradient algorithm for NE. Concluding remarks are in Section 7.

2 Polyhedral Convexity

This section introduces duality tools from convex analysis to give an overview of properties common to exchange and games.

2.1 Games

For $x = (x_1, \ldots, x_M), y = (y_1, \ldots, y_M) \in \mathbb{R}^M$, $y \cdot x = \sum_{r=1}^{M} y_r x_r$.

Elements $x^k \in \mathbb{R}^M, k = 1, \ldots, K$, define a bounded polyhedral convex set

$$C = \{x = \sum_k \lambda_k x^k : x^k \in C, \lambda_k \geq 0, \sum_k \lambda_k = 1\}. \tag{2.1.1}$$

C can be defined by its *indicator function* $\delta_C : \mathbb{R}^m \to \mathbb{R} \cup \{\infty\}$, where

$$\delta_C(x) = \begin{cases} 0 & \text{if } x \in C. \\ \infty, & \text{otherwise} \end{cases} \tag{2.1.2}$$

The data of a non-cooperative game will be defined by a convex polyhedron.

The *subdifferential* of δ_C at x are the $y \in \mathbb{R}^M$ such that

$$\partial\delta_C(x) = \{y : y \cdot x - \delta_C(x) \geq y \cdot x' - \delta_C(x'), \forall x'\} \tag{2.1.3}$$

$\partial\delta_C(x) \neq \emptyset$ implies $x \in C$. A non-empty subdifferential $\partial\delta_C(x)$ is an unbounded convex polyhedron $D = \{y : y = \sum_k \lambda_k y^k, \lambda_k \geq 0, y^k \in \partial\delta_C(x), \forall k\}$, where $y \in D$ implies $\lambda y \in D, \forall \lambda > 0$. $\partial\delta_C(x)$ will be normalized by restricting attention to

$$\partial\delta_C(x) \cap Y := \{y : |y| = \sum_m^M |y_m| \leq 1\} \tag{2.1.4}$$

Hence, $\partial\delta_C(x)$ is also bounded convex polyhedron in $\mathbb{R}^M$.

The *conjugate* of δ_C, known as the support function of C, is

$$\delta_C^*(y) = \sup_x \{y \cdot x - \delta_C(x)\} \tag{2.1.5}$$

that is convex in y. Its subdifferential is

$$\partial\delta_C^*(y) = \{x : y \cdot x - \delta_C(x)\}, \tag{2.1.6}$$

a non-empty polyhedral convex set that is necessarily bounded. From positive homogeneity of the support function, it suffices to restriction attention to $\partial\delta_C^*(y), y \in Y$.

From their definitions,

$$\delta_C^*(y) + \delta_C(x) \geq y \cdot x, \quad \forall y, x \in \mathbb{R}^M \tag{2.1.7}$$

called Fenchel's Inequality. It is readily established that there is equality if and only if

$$\delta^*_C(y) + \delta_C(x) = y \cdot x \Longleftrightarrow x \in \partial\delta^*_C(y) \text{ and } y \in \partial\delta_C(x) \tag{2.1.8}$$

The properties of $\partial\delta_C$ or $\partial\delta^*_C$ define the geometry of C. For $y \in Y$, $\partial\delta^*_C(y)$ is either a singleton or

$$\partial\delta^*_C(y) = F = \{x : x = \lambda_k \bar{x}^k, \bar{x}_k \in \partial\delta^*_C(y), k = 1, \ldots, K, \lambda_k \geq 0, \sum_k \lambda_k = 1\}$$

a polyhedral convex subset of C, called an *exposed face* of C. It satisfies

$$\delta^*_C(y) = y \cdot x \geq y \cdot C = \{y \cdot x : x \in C\} \tag{2.1.9}$$

Since the boundary of C consists of a finite number of exposed faces $\delta^*_C(y^h), h = 1 \ldots, H$, the geometry of C can be obtained as the intersection of the hyperplanes (affine functions) $(y^h, \delta^*_C(y^h)) \in \mathbb{R}^{M+1}$, where

$$C = \{x : \delta^*_C(y^h) \geq y^h \cdot x, h = 1, \ldots H\}. \tag{2.1.10}$$

known as the *external representation* of C.

For purposes of price-taking maximization, interest in C can be confined to

$$C^+ := \{x \in C : x \in \partial\delta^*_C(y), y \in \Delta^M\},$$

where

$$\Delta^M = Y \cap \mathbb{R}^M_+,$$

the non-zero, non-negative prices, normalized prices. I.e., knowlege of $\partial\delta^*_C(y^h)$ defining the exposed faces $y^h \in \Delta^M$ defines the relevant boundary of C.

When interest in C is confined to C^+, call its duality *complete* if C^+ is known. C^+ can be identified from knowledge of the finite subset

$$C^+ = \{x \in C : x \in \partial\delta^*_C(y^h), y^h \in \Delta^M\}$$

Say that C^+ has an *incomplete duality* with respect to $\mathfrak{T} \subset \Delta^M$ if its external representation requires elements in $\Delta^M \backslash \mathfrak{T}$. Incompleteness implies there is a pair $(y, x) \in \Delta^M \times C$ such that

$$\delta^*_C(y) + \delta_C(x) = y \cdot x > \delta^*_C(y') + \delta_C(x), \quad \forall y' \in \mathfrak{T} \tag{2.1.11}$$

Completeness holds at a *particular* (y, x) if equality is achieved.

Completeness is related to bi-conjugate duality. For polyhedral convexity (Rockafellar [1970, Theorem 12.2]):

$$\delta^{**}_C(x) = \sup_{y \in Y} \{\delta^*_C(y) - y \cdot x\} = \delta_C(x) \tag{2.1.12}$$

i.e., the conjugate of the conjugate is the function itself. Incompleteness is associated with the failure of bi-conjugacy:

$$\delta_C^{**}(x) = \sup_{y \in \mathfrak{T}} \{\delta_C^*(y) - y \cdot x\} < \delta_C(x) \tag{2.1.13}$$

Altrough completeness (of the relation between prices y and quantities x) is a desirable property, modeling assumptions may preclude it. For purposes of establishing existence of equilibrium, it need not be necessary. As an appropriate illustration,

$$\inf_{y' \in \mathfrak{T}} \delta_C^*(y') = \delta_C^*(y) + \delta_C(x) = y \cdot x = 0 \Longleftrightarrow 0 \geq y \cdot C \tag{2.1.14}$$

may also imply the same equality when the infimum is over $y \in \Delta^M$. In that case, differences in maximizing responses between complete and incomplete dualities would only be observed away from equilibrium.

Delegation in games will be described as the construction of another polyhedron C_d derived from C such that $C \subset C_d$. This permits $x \in C_d \backslash \{C\}$ and

$$\delta_{C_d}^*(y) + \delta_{C_d}(x) > \delta_C^*(y) + \delta_C(x)$$

since $\delta_C(x) = -\infty$. Evidently, an exposed face of C may lie within C_d.

The extension below will include a simplifying feature that $\mathbf{0} \in C_d$, not necessarily available in C. This will allow the equilibrium objective in (2.1.14) to be

$$\delta_{C_d}^*(y) + \delta_{C_d}(\mathbf{0}) = 0 \tag{2.1.15}$$

A remarkable feature (see Proposition 3.3.2) will be that although C_d is a superset of C. when price maximization is restricted is $\mathfrak{T}$,

$$y \in \mathfrak{T} \text{ and } \delta_C^*(y) + \delta_C(x) = y \cdot x \Longrightarrow \delta_{C_d}^*(y) + \delta_{C_d}(x) = y \cdot x, \tag{2.1.16}$$

i.e., opportunties for gain are not improved.

2.2 Exchange

A bounded polyhedral set for exchange will again be denoted by C, but in $\mathbb{R}^\ell$, to call attention to the difference in dimensonality. For exchange, $\mathbf{0} \in C$.

The indicator function δ_C is replaced by a real-valued function polyhedral concave function on C, $f : \mathbb{R}^\ell \to \mathbb{R} \cup \{-\infty\}$, Since f is concave, $-f$ is convex.

With an abuse of terminology, define the "conjugate" of f as

$$f^*(y) = \sup_{x'} \{f(x') - y \cdot x'\} \tag{2.2.1}$$

that is convex in y. Consistent with that abuse, define its subdifferential as

$$\partial f(x) = \{y : f(x) - y \cdot x \geq f(x') - y \cdot x'\}. \tag{2.2.2}$$

The Fenchel Inequality for (convex) f^* and (concave) f is

$$f^*(y) - f(x) \geq y \cdot x, \forall x, y \in \mathbb{R}^\ell \tag{2.2.3}$$

Similar to (2.1.8),

$$f^*(y) - f(x) = y \cdot x \Longleftrightarrow x \in \partial f^*(y) \longleftrightarrow y \in \partial f(x). \tag{2.2.4}$$

The analog of bi-conjugacy of δ_C (2.1.12) for f is

$$f^{**}(x) = \sup_y \{f^*(y) - y \cdot x\} = f(x) \tag{2.2.5}$$

Again, interest is only in

$$C^+ = \{x : x \in \partial f^*(y), y \in \mathbb{R}^\ell_+\}$$

Because f is polyhedral, there is a finite set $y^k \in \mathbb{R}^\ell_+$ such that

$$C^+ = \bigcup_{k \in K} \partial f^*(y^k) \tag{2.2.6}$$

Price-taking maximization on C^+ provides relevant information about f. From a refinement of Rockafellar's Subdifferentiability Theorem [1970, Theorem 24.9] by Kocourek [2010], the polyhedral function can be identified.

Proposition 2.2.1 *If f and f' are polyhedral concave functions defined on C^+ with exposed faces given by the same $\{y^h\} \subset \mathbb{R}^\ell_+$, i.e., $\partial f^*(y^h) = \partial f'^*(y^h)$, then $f'(x) = f(x) + \alpha, \quad \alpha \in \mathbb{R}$*

The counterpart to the complications caused by restricting prices to $\mathfrak{T}$ in games is a budget restriction in exchange. Although the sources of complications between games and exchange differ, their consequences are similar:

Define

$$f(x; y) = \begin{cases} f(x) & \text{if } y \cdot x \leq 0 \\ 0, \text{otherwise} & \end{cases} \tag{2.2.7}$$

to be called the non-QL budgetary restriction, the settingfor non-QL WE, below.

The counterpart of f^* is

$$f^*(y; 0) = \sup_x \{f(x; y)\} \tag{2.2.8}$$

Compared to $f^*(y)$ allowing all $x \in C$, $f^*(y; 0)$ permits only those for which $y \cdot x \leq 0$. $f^*(y; 0)$, known in economics as an *indirect utility function*, is *not* convex in y. (The set $\mathfrak{T}$ in Section 2.1, to be used in games, will not be convex in Δ^M.)

By construction,

$$f^*(y) \geq f^*(y; 0)$$

Therefore,

$$f^*(y) - f(x) = y \cdot x \geq f^*(y; 0) - f(x; y) \tag{2.2.9}$$

The inequality can be characterized as $f(x;y)$ imposing an incomplete duality with respect to maximization compared to f. As above, incompleteness need not preclude equilibrium.

With the budgetary restriction, it is readily established that prices in $\mathbb{R}^\ell_+$ exhibit the positive homogeneity of support functions:

$$f^*(\beta y;0) = f^*(y;0) \quad \forall \beta > 0; \tag{2.2.10}$$

i.e., maximizing choices depend only on relative prices. In comparison, $f^*(\beta y)$ varies with $\beta > 0$. It is well-known that such variations can be exploited to find β such that

$$\partial f^*(y;0) = \partial f^*(\beta y;0) = \partial f^*(\beta y) \tag{2.2.11}$$

This implies

$$[\beta^{-1} f]^*(y) = \beta^{-1} f^*(y;0) \tag{2.2.12}$$

In other words, $\alpha := \beta^{-1}$ induces the optimal choice x for $\alpha f(x) - y \cdot x$ and $f(x;y)$. The differences in maximizing responses would only be observed away from equilibrium.

3 Price-taking Maximization

3.1 Activity Analysis Version of an Exchange Economy $\mathcal{E}$

This section describes a model of exchange in a setting resembling the data defining a normal form game.

The characteristics of individual i are given by $(A_i, \dot{E}_i, \dot{\nu}_i)$. A_i is a finite index set of activities available to i. $\dot{E}_i \in \mathbb{R}^\ell \times \mathbb{R}^{A_i}$ is a matrix with elements

$$\dot{e}_i(c)[b_i] \tag{3.1.1}$$

with $c = 1, \ldots, \ell$ representing commodities and $b_i \in A_i$. The commodity vector representing activity b_i is

$$\dot{e}_i(b_i) \in \mathbb{R}^\ell, \tag{3.1.2}$$

where positive elements are demands and negative elements are supplies. The utility of activity b_i is $\dot{\nu}_i(\dot{e}_i(b_i))$.

Commodity vectors $\dot{e}_i[b_i]$ are indivisible. Divisibility is obtained via convexification as

$$E_i := \{e_i = \dot{E} z_i = \sum_{b_i} \dot{e}_i[b_i] z_i[b_i] : z_i \in \Delta(A_i)\} \subset \mathbb{R}^\ell \tag{3.1.3}$$

where

$$\Delta(A_i) := \{z_i : A_i \to \mathbb{R} : z_i[b_i] \geq 0, \forall b_i \in A_i, \sum_{b_i} z_i[b_i] = 1\}$$

is the unit simplex associated with A_i. Hence, E_i is a bounded polyhedral convex set in $\mathbb{R}^\ell$.

The *concave* indicator function of E_i is

$$\delta_{E_i}(e'_i) = \begin{cases} 0 & \text{if } e'_i \in E_i, \\ \infty & \text{otherwise} \end{cases}$$

From the characteristics $(\dot{E}_i, \dot{\nu}_i)$, define

$$\begin{aligned} \dot{\nu}_i(e_i(z_i)) &:= \sum_{b_i} \dot{\nu}_i(\dot{e}_i[b_i]) z_i[b_i] - \delta_{E_i}(e_i(z_i)) \\ &\leq \sup_{z_i} \left\{ \dot{\nu}_i(e_i(z_i)) : e_i(z_i) = e_i \right\} - \delta_{E_i}(e_i) := \nu_i(e_i) \end{aligned} \tag{3.1.4}$$

I.e., ν_i is the smallest polyhedral concave function derived from $(\dot{E}_i, \dot{\nu}_i)$.

Assume

$$(i): \quad \mathbf{0} \in \dot{E}_i \qquad (ii): \quad e_i, e'_i \in E_i, e_i \geq e'_i, e_i \neq e'_i, \nu_i(e_i) > \nu_i(e'_i)$$

A model of an exchange economy $\mathcal{E}$ is defined by $\langle \nu_i \rangle$ summarizing the characteristics $(A_i, \dot{E}_i, \dot{\nu}_i)$ of n individuals.

Remark 1: Do-it-yourself concavification of ν_i is similar to individual convexification in a game. In a more standard presentation, concavity of ν_i would be assumed: An implicit index set $A_i \subset [0, \infty)$ could define convexity of E_i directly, where each $\dot{e}_i[b_i], b_i \in [0, 1]$ is an extreme point of E_i, along with the assumption that ν_i is concave on E_i, or more generally quasi-concave. When A_i is finite, quasi-concavity can be replaced by concavity. (Fenchel [1951])

3.1.1 Quasi-Linear (QL) Utility

Utility is *quasi-linear* (QL) for i if the utility of $(e_i, m_i) \in \mathbb{R}^\ell \times \mathbb{R}$ is $\nu_i(e_i) + m_i$. At prices p for e_i and *fixed* price 1 for m_i, i's budget constraint is $p \cdot e_i + m_i = 0$. The *indirect utility function* for ν_i is the maximum utility i can achieve at p,

$$\nu_i^*(p) = \sup_{e'_i} \{\nu_i(e'_i) - p \cdot e'_i\} = \max_{b_i} \{\dot{\nu}_i(\dot{e}_i[b_i]) - p \cdot \dot{e}_i[b_i]\} \geq 0 \tag{3.1.5}$$

Concavity of ν_i in e_i implies that ν_i^* is convex in p. Therefore, $-\nu_i^*(p)$ is concave. The convex version of the Fenchel Inequality for ν_i and ν_i^* is

$$\nu_i^*(p) - \nu_i(e_i) \geq -p \cdot e_i, \quad \forall p, e_i \in \mathbb{R}^\ell \tag{3.1.6}$$

Rather than pairing $-\nu_i^*$ with ν_i, ν_i^* will be used because it is a more direct description of utility maximization.

The *subdifferential* of ν_i^* at p,

$$\partial\nu_i^*(p) = \{e_i : (p' - p)\cdot e_i \geq \nu_i^*(p') - \nu_i^*(p), \forall p' \in \mathbb{R}^\ell\} \tag{3.1.7}$$

defines utility maximizing choices, i.e., the *excess demand correspondence* of ν_i at p.

The concave version of the *subdifferential* of ν_i at e_i is

$$\partial\nu_i(e_i) = \{p : p\cdot(e_i' - e_i) \geq \nu_i(e_i') - \nu_i(e_i), \forall e_i' \in \mathbb{R}^\ell\} \tag{3.1.8}$$

[If $\partial\nu_i(e_i)$ is a singleton, $p = \nabla\nu_i(e_i)$ is the vector of marginal utilities.]

A well-known and readily established result is:

Proposition 3.1.1 *The following are equivalent:*

$$(\bullet) \quad -\nu_i^*(p) + \nu_i(e_i) = p\cdot e_i \quad (\bullet) \quad e_i \in \partial\nu_i^*(p) \quad (\bullet) \quad p \in \partial\nu_i(e_i)$$

3.1.2 From QL to non-QL Utility

Utility is *non-quasi-linear* (non-QL) if there is no money commodity. The budget constraint $p\cdot e_i + m_i = 0$ becomes $p\cdot e_i \leq 0$. Let

$$\nu_i(e_i\,|\,p) = \begin{cases} \nu_i(e_i) & \text{if } p\cdot e_i \leq 0, \\ -\infty & \text{if } p\cdot e_i > 0 \end{cases} \tag{3.1.9}$$

The conjugate function $\nu_i^*(p) = \sup_{e_i'}\{\nu_i(e_i') - p\cdot e_i'\}$ is replaced by the *indirect utility function*

$$\nu_i^*(p;0) := \sup_{e_i'}\{\nu_i(e_i'\,|\,p)\} \tag{3.1.10}$$

The counterpart of $\partial\nu_i^*(p)$ is the non-QL demand correspondence

$$\widetilde{\partial}\,\nu_i^*(p;0) := \{e_i : \nu_i(e_i') > \nu_i(e_i) \Longrightarrow p\cdot e_i' > 0\} \tag{3.1.11}$$

Differences between the QL and non-QL indirect utility functions are:

$$\nu_i^*(\cdot) \text{ is convex in } p; \qquad \nu_i^*(\cdot;0) \text{ is quasi-convex in } p. \tag{3.1.12}$$

And for $\alpha > 0$,

$$[\alpha\nu_i]^*(p) \geq \alpha\nu_i^*(p;0) \tag{3.1.13}$$

reflecting the fact that the non-QL budget constraint is a restriction of QL.

With no money commodity, $\alpha\nu_i$ is similar to ν_i. Consequently, in comparison to (1.1.13)

$$[\alpha\nu_i]^*(p;0) = \alpha\nu_i^*(p;0) \tag{3.1.14}$$

from which it follows that

$$\widetilde{\partial}\,[\alpha\nu_i]^*(p;0) = \widetilde{\partial}\,\nu_i^*(p;0) = \widetilde{\partial}\,\nu_i^*(\alpha^{-1}p;0) \tag{3.1.15}$$

Nevertheless, at a given p, the QL and non-QL models of utility maximization can be made to coincide.

For QL, $\alpha\nu_i, \alpha \neq 1$, changes the utility trade-off between the money and non-money compared to ν_i. It is well-known that by modifying the substitution parameter α *at a given* p, the non-QL constraint need not be binding.

Proposition 3.1.2 *For $p \in \mathbb{R}^\ell_+$ and ν_i, there exist $\bar{\alpha} > 0$ such that*

$$[\bar{\alpha}\nu_i]^*(p) = \bar{\alpha}\nu_i^*(p;0) \text{ and } \partial\,[\bar{\alpha}\nu_i]^*(p) = \widetilde{\partial}\,\bar{\alpha}^{-1}p \in \partial\nu_i(e_i) \quad \bar{\alpha}^{-1}p \cdot e_i = 0$$

From the conjugate duality between $\bar{\alpha}\nu_i$ and $[\bar{\alpha}\nu_i]^*$, the following are equivalent:

$$(\bullet) \quad e_i \in \partial[\bar{\alpha}\nu]_i^*(p) \quad (\bullet) \quad p \in \partial\,\bar{\alpha}\nu_i(e_i) \quad (\bullet) \quad \bar{\alpha}^{-1}p \in \partial\nu_i(e_i)$$

Thus, for a suitable choice $\bar{\alpha}$, there exists $\bar{\alpha}^{-1}p \in \partial\nu_i(e_i)$ for the QL problem satisfying the non-QL restriction $[\bar{\alpha}]^{-1}p \cdot e_i = 0$.

Remark 2: (Indirect utility with and without conjugate duality) The conjugate of the concave function $-\nu_i^*$, called the bi-conjugate of ν_i, exhibits the well-known result that

$$-\nu_i^{**}(e_i) = \inf_p \{p \cdot e_i - [-\nu_i^*(p)]\} = \inf_p \{p \cdot e_i - [\inf_{e_i} \{p \cdot e_i - \nu_i(e_i)\}]\} = \nu_i(e_i) \qquad (3.1.16)$$

I.e., the conjugate of the conjugate is the function itself. This pinpoints an essential difference with the non-QL model where

$$-\nu_i^{**}(e_i;0) = \inf_p \{p \cdot e_i - [-\nu_i^*(p;0)]\} \leq \nu_i(e_i), \qquad (3.1.17)$$

since by (1.1.13), $\nu_i^*(p) \geq \nu_i^*(p;0)$. Failure of conjugate duality with non-QL utility occurs because the non-QL budget constraint reduces utility gains below what is achievable with the the QL constraint.

3.2 Exchange Representations of a Game $\mathcal{G}$

This section describes a normal form game in terms resembling the data defining a model of exchange.

The characteristics of individual i are given by $(A, \mathfrak{u}_i)$ where

$$A = A_1 \times A_2 \times \cdots \times A_n$$

are *joint activities* and $\mathfrak{u}_i : A \to \mathbb{R}$. For fixed A, a game $\mathcal{G}$ is defined by $\langle \mathfrak{u}_i \rangle$.

In $\mathcal{E}$ the consequences of activity b_i requires its prior translation into a vector of commodity excess demands, $\dot{e}_i[b_i]$, to define the matrix of utility gains $\dot{E}_i = \langle \dot{\nu}_i(\dot{e}_i(c)[b_i]) \rangle$. In $\mathcal{G}$ a commodity representation of an activity defines a matrix of *individual deviation gains* or a matrix of *delegated deviation gains* defined below.

3.2.1 Individual Deviation Gains

The matrix $\dot{\mathfrak{E}}_i(\mathfrak{u}_i) \in \mathbb{R}^A \times \mathbb{R}^{A_i}$ with elements

$$\dot{\mathfrak{e}}_i(\mathfrak{u}_i)(a)[b_i] = \mathfrak{u}_i(b_i, a_{-i}) - \mathfrak{u}_i(a_i, a_{-i}) \in \mathbb{R},$$

is the *deviation gain* from choosing the activity $b_i \in A_i$ at a; and $\dot{\mathfrak{e}}_i(\mathfrak{u}_i)[b_i] = \langle \mathfrak{e}_i(a)[b_i] \rangle \in \mathbb{R}^A$ is the *vector* of those gains over all $a \in A$. When $\mathfrak{u}_i$ is fixed, explicit dependence of deviation gains will suppressed by writing $\dot{\mathfrak{e}}_i[b_i]$.

The vector $\dot{\mathfrak{e}}_i[b_i] \in \mathbb{R}^A$ in a game will play the role of $\dot{e}_i[b_i] \in \mathbb{R}^\ell$ in exchange. [Note: Information contained in the *vector* of deviation gains from $\mathfrak{u}_i$ effectively incorporates information from $\dot{e}_i[b_i]$ along with $\dot{\nu}_i(\dot{e}_i[b_i])$ in the model of exchange.] As in $\dot{E}_i$ for exchange, convex combinations of the columns of $\dot{\mathfrak{E}}_i$ yield

$$\mathfrak{E}_i = \{\mathfrak{e}_i = \dot{\mathfrak{E}}_i z_i = \sum_{b_i} \dot{\mathfrak{e}}_i[b_i] z_i[b_i] : z_i \in \Delta(A_i)\} \subset \mathbb{R}^A$$

By construction, deviation gains are monotone; i.e., if $\mathfrak{e}_i, \mathfrak{e}'_i \in \mathfrak{E}_i$ and $\mathfrak{e}_i \geq \mathfrak{e}'_i$, $\mathfrak{e}_i$ is at least as good as $\mathfrak{e}'_i$.

3.2.2 Delegated Deviation Gains

The definition of $\dot{\mathfrak{E}}_i$ assumes each i can chose only one $b_i \in A_i$. A modification allows i to delegate to each a_i — as if acting as a separate agent of i. For $a_i \in A_i$, the possible deviation gains are — suppressing its explicit dependence on $\mathfrak{u}_i$,

$$\dot{\mathfrak{e}}_{a_i}(a')[b_i] = \begin{cases} \mathfrak{u}_i(b_i, a'_{-i}) - \mathfrak{u}_i(a_i, a'_{-i}) & \text{if } a'_i = a_i, \\ 0 & \text{if } a'_i \neq a_i \end{cases}$$

i.e., when $a'_i \neq a_i$, a_i is inactive. Just as $\dot{\mathfrak{e}}_i[b_i] \in \mathbb{R}^A$, $\dot{\mathfrak{e}}_{a_i}[b_i] = \langle \mathfrak{e}_{a_i}(a')[b_i] \rangle \in \mathbb{R}^A$ is column b_i of the matrix $\dot{\mathfrak{E}}_{a_i} = \langle \dot{\mathfrak{e}}_{a_i}[b_i] \rangle \in \mathbb{R}^A \times \mathbb{R}^{A_i}$. Convexification is defined as above by

$$\mathfrak{E}_{a_i} = \{\mathfrak{e}_{a_i} = \dot{\mathfrak{E}}_{a_i} z_{a_i} = \sum_{b_i} \dot{\mathfrak{e}}_{a_i}[b_i] \mathfrak{z}_{a_i}[b_i] : \mathfrak{z}_{a_i} \in \Delta(A_i)\} \subset \mathbb{R}^A$$

Delegation implies that i's deviation gain possibilities are $\dot{\mathfrak{E}}_{A_i} = \langle \dot{\mathfrak{E}}_{a_i} \rangle$, a matrix with rows indexed by A rows and columns by $A_i \times A_i$.

$$\mathfrak{E}_{A_i} = \sum_{a_i} \mathfrak{E}_{a_i} = \{\mathfrak{e}_{A_i} = \dot{\mathfrak{E}}_{A_i} z_{A_i} = \sum_{a_i} \Big(\sum_{b_i} \dot{\mathfrak{e}}_{a_i}[b_i] \mathfrak{z}_{a_i}[b_i] \Big) : \mathfrak{z}_{a_i} \in \Delta(A_i), a_i \in A_i\}$$

Delegation makes possible the choice

$$\mathfrak{z}^{\mathrm{ID}}_{a_i} = (\mathfrak{z}_{a_i}[a_i] = 1, \mathfrak{z}_{a_i}[b_i] = 0, b_i \neq a_i),$$

allowing each a_i the option not to deviate no matter what the value of $a' \in A$, with the result that $\mathfrak{e}_{a_i}(\mathfrak{z}^{\text{ID}}_{a_i}) = \mathbf{0} \in \mathfrak{E}_{a_i}$. I.e., delegation includes the no-trade option.

Remark 3: (Activities underlying Choices) Each of the matrices $\dot{E}_i$ in exchange and $\dot{\mathfrak{E}}_i$, $\dot{\mathfrak{E}}_{a_i}$ in games have columns indexed by $b_i \in A_i$. The rows of $\dot{E}_i$ in exchange are indexed $c = 1, \ldots, \ell$, and the rows of $\dot{\mathfrak{E}}_i, \dot{\mathfrak{E}}_{a_i}$ by $a \in A$. The same $\Delta(A_i)$ defining convex combinations of columns converts these matrices into convex sets $E_i \subset \mathbb{R}^\ell$, and $\mathfrak{E}_i, \mathfrak{E}_{a_i} \subset \mathbb{R}^A$.

Remark 4: (Strategic Equivalence as Deviation Gains) It is well-known that if instead of $\mathfrak{u}_i$, the individual's utility were $\mathfrak{u}'_i = \mathfrak{u}_i + \mathfrak{w}_i$, where $\mathfrak{w}_i : A_{-i} \to \mathbb{R}$ is an arbitrary function, then

$$\mathfrak{u}'_i(b_i, a_{-i}) - \mathfrak{u}'_i(b_i, a_{-i}) = \mathfrak{u}_i((b_i, a_{-i}) + \mathfrak{w}_i(a_{-i}) - [\mathfrak{u}_i(a_i, a_{-i}) + \mathfrak{w}_i(a_{-i}) = \mathfrak{e}_i(a)[b_i] \quad (3.2.1)$$

A similar equality applies when there is delegation. With respect to non-cooperative behavior there is strategic equivalence: a "lump-sum" change in i's utility associated with the behavior of others would not affect i's payoffs with respect to changing from a_i to b_i. This feature distinguishes non-cooperative from cooperative games, where $\mathfrak{u}'_i$ would not typically be regarded as equivalent to $\mathfrak{u}_i$.

3.2.3 Restricted and Unrestricted Prices

The same notation for prices in exchange, where $p \in \mathbb{R}^\ell$, will be used in games, with $p \in \mathbb{R}^A$, where $p(a)$ is the price of a. Differences in the dimensions of terms $e_i \in \mathbb{R}^\ell$ accompanying prices in exchange and $\mathfrak{e}_i, \mathfrak{e}_{a_i} \in \mathbb{R}^A$ in games will discriminate between them.

A normalization of non-negative, non-zero prices in $\mathbb{R}^A$ is

$$P := \Delta(A) = \{p : p(a) \geq 0, \forall a, \sum_a p(a) = 1\}$$

This definition has $p \in \mathbb{R}^A$ which has the same dimension, $|A| = |A_1| \times |A_2| \times \cdots \times |A_n|$, as $\mathfrak{u}_i$.

In contrast, the set

$$Q := \times_i \Delta(A_i) \subset \mathbb{R}^{\sum_i |A_i|}$$

is used to define prices for NE. The *tensor* mapping, $\mathfrak{T} : Q \to P$, of Q where

$$\mathfrak{T}(q) = \bigotimes_i q_i$$

translates $q = (q_1, \ldots, q_n) \in \times\Delta(A_1) \times \ldots \times \Delta(A_n)$ into its representation in P. The tensor mapping regards prices/independent probabiliites as taking place in $\mathbb{R}^A$, where P is the default normalization and $\mathfrak{T}[Q]$ is a restriction.

3.3 Maximization in Games

Demand correspondences in exchange vary with QL and non-QL budget constraints. Demand correspondences in games vary with respect to deviation gain opportunities and with respect to allowable prices.

3.3.1 Demands without delegation

For non-delegated choice, the functional representation of i's constraints is the indicator function of $\mathfrak{E}_i$,$\delta_{\mathfrak{E}_i} \in \{0, \infty\}$. The analog of the indirect utility function in exchange is the conjugate function

$$\delta^*_{\mathfrak{E}_i}(p) := \sup_{\mathfrak{e}'_i} \{p \cdot \mathfrak{e}'_i - \delta_{\mathfrak{E}_i}(\mathfrak{e}'_i)\} = \max_{b_i} \{p \cdot \dot{\mathfrak{e}}_i[b_i]\}, \tag{3.3.1}$$

where $p \in P$. The convex version of the Fenchel Inequality corresponding to (1.1.7) in exchange is

$$\delta^*_{\mathfrak{E}_i}(p) + \delta_{\mathfrak{E}_i}(\mathfrak{e}_i) \geq p \cdot \mathfrak{e}_i, \quad \forall p, \mathfrak{e}_i \in \mathbb{R}^A \tag{3.3.2}$$

The demand correspondence at p is the subdifferential of $\delta^*_{\mathcal{E}_i}(p)$,

$$\partial\delta^*_{\mathfrak{E}_i}(p) = \{\mathfrak{e}_i : (p' - p) \cdot \mathfrak{e}_i \geq \delta^*_{\mathfrak{E}_i}(p') - \delta^*_{\mathfrak{E}_i}(p), \forall p'\} \tag{3.3.3}$$

The subdifferential of $\delta_{\mathfrak{E}_i}(\mathfrak{e}_i)$ is

$$\partial\delta_{\mathfrak{E}_i}(\mathfrak{e}_i) = \{p : p \cdot (\mathfrak{e}'_i - \mathfrak{e}_i) \geq \delta_{\mathfrak{E}_i}(\mathfrak{e}'_i) - \delta_{\mathfrak{E}_i}(\mathfrak{e}_i).\forall \mathfrak{e}'_i\} \tag{3.3.4}$$

The counterpart of Proposition (3.1.1) in exchange is

Proposition 3.3.1 *The following are equivalent:*

$$(\bullet) \quad \delta^*_{\mathfrak{E}_i}(p) + \delta_{\mathfrak{E}_i}(\mathfrak{e}_i) = p \cdot \mathfrak{e}_i \quad (\bullet) \quad \mathfrak{e}_i \in \partial\delta^*_{\mathfrak{E}_i}(p) \quad (\bullet) \quad p \in \partial\delta_{\mathfrak{E}_i}(\mathfrak{e}_i)$$

Since $\mathfrak{T}[Q] \subset \mathbb{R}^A$, each of the above statements applies when $p = \mathfrak{T}(q)$.

3.3.2 Demands with delegation

When i can delegate to each a_i, the indicator function representing choices available to i is

$$\delta_{\mathfrak{E}_{A_i}}(\mathfrak{e}) := \inf \left\{ \sum_{a_i} \delta_{\mathfrak{E}_{a_i}}(\mathfrak{e}_{a_i}) : \mathfrak{e} = \sum_{a_i} \mathfrak{e}_{a_i} \right\}$$

where $\delta_{\mathfrak{E}_{a_i}}$ is the indicator of $\mathfrak{E}_{a_i}$. Evidently $\mathfrak{E}_i \subseteq \mathfrak{E}_{A_i}$.

With delegated choice, each $a_i \in A_i$ can be regarded as a price-taker whose objective is

$$\delta^*_{\mathfrak{E}_{a_i}}(p) = \sup_{\mathfrak{e}'_{a_i}} \{p \cdot \mathfrak{e}'_{a_i} - \delta_{\mathfrak{E}_{a_i}}(\mathfrak{e}'_{a_i})\} \tag{3.3.5}$$

Recall that with delegation, $\mathfrak{e}_{a_i}(a')[b_i] = 0$, whenever $(a_i', a_{-i}), a_i' \neq a_i$. For delegated choice, the support of p defines who gets to choose:

$$p(a_i) = \sum_{a_{-i}} p(a_i, a_{-i}) = 0 \Longrightarrow p \cdot \mathfrak{e}_{a_i} = 0. \tag{3.3.6}$$

And since $\sum_{a_i}[p(a_i) = \sum_{a_{-i}} p(a_i, a_{-i})] = 1$, the weighted sum of over A_i is the same as non-delegation. Consequently,

$$\delta^*_{\mathfrak{E}_{A_i}}(p) := \sup_{e'} \left\{p \cdot \mathfrak{e}' - \delta_{\mathfrak{E}_{A_i}}(\mathfrak{e}')\right\} = \sum_{a_i} \sup_{\mathfrak{e}_{a_i}} \left\{p \cdot [\mathfrak{e}'_{a_i} - \delta_{\mathfrak{E}_{a_i}}(\mathfrak{e}'_{a_i})]\right)\right\} = \sum_{a_i} p(a_i)\delta^*_{\mathfrak{E}_{a_i}}(p), \tag{3.3.7}$$

illustrates the decentralization property of price-taking maximization: acting independently, each $a_i \in A_i$ can do as well as if they acted jointly.

Since delegated choices are a superset of non-delegated,

$$\delta^*_{\mathfrak{E}_{A_i}}(p) \geq \delta^*_{\mathfrak{E}_i}(p) \tag{3.3.8}$$

However, when prices are restricted to $\mathfrak{T}[Q]$, the variety (correlation) of prices implies that the benefits of delegation disappear.

Proposition 3.3.2 *For* $p = \bigotimes_i q_i \in \mathfrak{T}[Q]$, $p(a_i) = \sum_{a_{-i}} \bigotimes_i q_i(a_i)q_{-i}(a_{-i}) = q_i(a_i)$,

$$\delta^*_{\mathfrak{E}_{A_i}}(\mathfrak{T}(q)) = \sum_{a_i} \delta^*_{\mathfrak{E}_{a_i}}(\mathfrak{T}(q)) = \delta^*_{\mathfrak{E}_i}(\mathfrak{T}(q)).$$

Remark 5: (Conjugate duality in games) Conjugacy in games exhibits the elementary duality between the indicator function of a bounded polyhedral convex set and its support function. As the subdifferentials of indicator functions, $\partial\delta_{\mathcal{E}_i}(\mathfrak{e}_i)$ and $\partial\delta_{\mathcal{E}_{a_i}}(\mathfrak{e}_{a_i})$ are normal cones. Consequently, each of the conjugates is positively homogeneous in p, e.g., $\delta^*_{\mathcal{E}_i}(\lambda p) = \lambda\delta^*_{\mathcal{E}_i}(p), \lambda > 0$; and its maximizing choices are homogeneous of degree zero in p, e.g., $\mathfrak{e}_i \in \partial\delta^*_{\mathcal{E}_i}(p)$ if and only if $\mathfrak{e}_i \in \partial\delta^*_{\mathcal{E}_i}(\lambda p), \lambda > 0$. Therefore, $\mathfrak{T}[Q] \subset P \subset \mathbb{R}^A_+$ can be regarded as an appropriate set to price deviation gains, just as $P^\ell = \{p : p(c) \geq 0, \sum_c p(c) = 1\} \subset \mathbb{R}^\ell_+$ prices commodities in non-QL exchange and $(\mathbb{R}^\ell_+, 1)$ prices commodities in QL exchange exchange.

The fact that $\delta_{\mathfrak{E}_i}(\mathfrak{e}_i), \delta_{\mathfrak{E}_{a_i}}(\mathfrak{e}_{a_i})$ and their conjugates $\delta^*_{\mathfrak{E}_i}(p), \delta^*_{\mathfrak{E}_{a_i}}(p)$ are convex when $p \in P$ implies the biconjugate properties

$$\delta^{**}_{\mathfrak{E}_i}(\mathfrak{e}_i) = \sup_p \{p \cdot \mathfrak{e}_i - \delta^*_{\mathfrak{E}_i}(p)\} = \delta_{\mathfrak{E}_i}(\mathfrak{e}_i) \qquad \delta^*_{\mathfrak{E}_{a_i}}(\mathfrak{e}_{a_i}) = \sup_p \{p \cdot \mathfrak{e}_{a_i} - \delta^*_{\mathfrak{E}_{a_i}}(p)\} = \delta_{\mathfrak{E}_{a_i}}(\mathfrak{e}_{a_i}) \tag{3.3.9}$$

In contrast, when p is constrained to $\mathfrak{T}[Q]$,

$$\delta^{**}_{\mathfrak{E}_i}(\mathfrak{e}_i) = \delta_{\mathfrak{E}_i}(\mathfrak{e}_i) = \sup_p \{p \cdot \mathfrak{e}_i - \delta^*_{\mathfrak{E}_i}(p)\} \geq \sup_q \{\mathfrak{T}(q) \cdot \mathfrak{e}_i - \delta^*_{\mathfrak{E}_i}(\mathfrak{T}(q))\}, \tag{3.3.10}$$

the counterpart of (3.1.18). I.e., the restriction on prices imposed by $\mathfrak{T}[Q]$ has the same consequences for bi-conjugate duality in games as the imposition of the non-QL budget constraint in exchange.

3.3.3 Zero Values for Individual Conjugates

A zero value for an indirect utility function in exchange implies that prices are such that the individual does not want to trade — a conclusion typically incompatible with equilibrium. With one exception, a zero value for the conjugate function of an individual will characterize equilibrium in games.

With delegation, since $\mathfrak{e}_{a_i}^{\text{ID}} = \mathbf{0} \in \mathfrak{E}_{a_i}$, $p \cdot \mathfrak{e}_{a_i}^{\text{ID}} = 0$ and therefore

$$\delta^*_{\mathfrak{E}_{a_i}}(p) \geq 0, \quad \forall p \in P \tag{3.3.11}$$

Without delegation, if $p = \mathfrak{T}(q) = \underset{i}{\otimes} q_i$, the choice $\mathfrak{e}_i(z_i) \in \mathfrak{E}_i$ for $z_i = q_i$ means i is accepting existing prices; hence, is effectively agreeing not to trade, with the result that $\mathfrak{T}(q) \cdot \mathfrak{e}_i(q_i) = 0$. Because this option is always available,

$$\delta^*_{\mathfrak{E}_i}(\mathfrak{T}(q)) \geq 0, \quad \forall q \in Q \tag{3.3.12}$$

However, without delegation when prices include $P \backslash \mathfrak{T}[Q]$, the individual loses the no-trade option. As a consequence there can exist $p \in P$,

$$\delta^*_{\mathfrak{E}_i}(p) < 0 \tag{3.3.13}$$

Example 1: (Negative deviation gain) Consider the following 2×2 in which for each outcome, both individuals receive the same utility, an identical interest game (Section 5.1).

	L	R
U	1,1	0,0
D	0,0	1,1

Consider probability distribution $\langle p(U, L) = \frac{2}{3}, p(U, R) = 0, p(DL) = 0, p(D, R) = \frac{1}{3} \rangle$. Since the marginal distributions are $\langle p_1(U) = \frac{2}{3}, p_1(D) = \frac{1}{3} \rangle$ and $\langle p_2(L) = \frac{2}{3}, p_2(R) = \frac{1}{3} \rangle$, $z_1 = \mathbf{1}_U$ and $z_2 = \mathbf{1}_L$ are clearly the best deviation strategies. Excess demands for U and L are:

$$\dot{e}_1[U] = (\dot{e}_1(U, L)[U], \dot{e}_1(U, R)[U], \dot{e}_1(D, L)[U], \dot{e}_1(D, R)[U]) = (0, 0, 1, -1),$$
$$\dot{e}_2[L] = (\dot{e}_2(U, L)[L], \dot{e}_2(U, R)[L], \dot{e}_2(D, L)[L], \dot{e}_2(D, R)[L]) = (0, 1, 0, -1).$$

The deviation gains, computed below, are strictly negative.

$$p \cdot \dot{e}_1 \mathbf{1}_U = (2/3, 0, 0, 1/3) \cdot (0, 0, 1, -1) = -1/3 < 0,$$
$$p \cdot \dot{e}_2 \mathbf{1}_L = (2/3, 0, 0, 1/3) \cdot (0, 1, 0, -1) = -1/3 < 0.$$

A no-trade/opt-out option can be introduced protecting the individual from negative gains by defining

$$\mathfrak{E}_i^0 = \mathfrak{E}_i \cup \{\mathbf{0}\} \tag{3.3.14}$$

Alternatively, the same result could be achieved by changing $\Delta(A_i)$ to $\Delta^0(A_i) = \{z_i : z_i(a_i) \geq 0, \sum_{a_i} z_i(a_i) \leq 1\}$. With $\mathfrak{E}_i^0$, individually maximized deviation gains are non-negative. With $\mathfrak{E}_i$, their sum can be negative.

Remark 6: (Resolving ambiguities in demand correspondences) Existence of equilibrium in games has raised the question: how to know, among multiple possibilities, which one to choose. For NE,

> On the face of it, equilibrium points in mixed strategies are unstable because any player can deviate without penalty from his equilibrium strategy even if all other players stick to theirs. (He can shift to any pure strategy to which his mixed equilibrium strategy assigns a positive probability; he can also shift to any arbitrary probability mixture of these pure strategies.) This instability seems to pose a serious problem because many games have only mixed-strategy equilibrium points. (Harsanyi [1973])

By reframing so that all individuals respond to the *same* prices in P, ambiguity with respect to non-cooperative equilibrium choices can be resolved. For NE, the condition $\delta^*_{\mathcal{E}_i}(\mathfrak{T}(q)) = 0$ means that although i could choose any z_i such that $\mathfrak{e}_i(z_i) \in \partial\delta^*_{\mathcal{E}_i}(\mathfrak{T}(q))$, i could adopt the convention that when prices are such deviation gains are 0, i agrees to make no change i.e., to choose $z_i = q_i$. For CE, when prices are such that a_i has an opportunity to choose, i.e., $p(a_i) > 0$, and $\delta^*_{\mathcal{E}_{a_i}}(p) = 0$ and therefore $e^{\mathrm{ID}}_{a_i} \in \partial\delta^*_{\mathcal{E}_{a_i}}(p) = 0$, a_i agrees to make no change. If unrestricted pricing were modeled with $\mathfrak{E}_i^{\mathbf{0}}$, then i can choose $\mathbf{0}$ when $\delta^*_{\mathfrak{E}_i^0}(p) = 0$. However, when individual choices are confined to $\mathfrak{E}_i$ and prices are in $P\backslash\{\mathfrak{T}[P]\}$, there is no *a priori* requirement for equilibrium deviation gains. Hence, there is no predetermined criterion upon which to resolve multiplicity of choices.

"Mixed-strategies" $e_i(z_i)$ also appear in exchange as a consequence of polyhedral concavity, leading to QL or non-QL demand correspondences, $\partial\nu_i^*(p)$ or $\widetilde{\partial}\,\nu_i^*(p;0)$. However, unlike games in which equilibrium occurs when $\delta_{\mathfrak{E}_i}(\mathfrak{T}(q)) = 0$ or $\delta_{\mathfrak{E}_{a_i}}(p) = 0$, ambiguities are more problematic in exchange because equilibrium values of $\nu_i^*(p)$ or $\widetilde{\partial}\,\nu_i^*(p;0)$ are not known. WE *ignores* the ambiguity associated with demand correspondences by defining equilibrium as existing if there is a $p \in \mathbb{R}^\ell$ such that *there exists a selection* from individual excess demands that clears the market.

4 Characterizations and Existence of Equilibria

Equilibrium is characterized by minimax/saddle-point conditions for WE in exchange and non-cooperative equilibrium in games. The minimax construction is based on opposition between individuals making choices to maximize their respective gains at given prices and a "price-maker" whose goal is to minimize those gains. Since maximization is measured by the values of the appropriate conjugate functions, the minimax value is defined by the minimum value of their sum. In exchange, minimization of maximum *utility* gains occurs only in the aggregate. In games, minimization of maximum *deviation* gains takes place at the individual level.

Emphasis on common characterization calls attention to differences required for the various minimax conditions. Existence of saddle-points, either by conjugate duality when relevant conditions are satisfied, or by appeal to a fixed-point argument when conjugate functions are not convex, is highlighted. The distinction applies both to exchange and games.

4.1 Exchange

4.1.1 QL WE

The price of the money commodity is fixed at 1. Relative prices of the non-money commodity are in $\mathbb{R}^\ell_+$ defined by the concave indicator function $\delta_{\mathbb{R}^\ell_+}(p) \in \{0, -\infty\}$, with conjugate

$$\delta^*_{\mathbb{R}^\ell_+}(e) = \inf_p \{p \cdot e - \delta_{\mathbb{R}^\ell_+}(p)\} = \begin{cases} 0 & \text{if } e \in \mathbb{R}^\ell_+, \\ -\infty, & \text{otherwise} \end{cases}$$

Aggregate utility and conjugate functions are

$$\nu_{\mathcal{E}}(e) = \sup \{\sum_i \nu_i(e_i) : \sum_i e_i = e\} \quad \nu^*_{\mathcal{E}}(p) = \sum_i \nu^*_i(p)$$

The saddle function is

$$S_{\mathcal{E}}(p, e) = \nu_{\mathcal{E}}(e) - p \cdot e - \delta_{\mathbb{R}^\ell_+}(p)$$

with minimax inequality

$$\inf_p \sup_e S_{\mathcal{E}}(p, e) = \inf_p \{\nu^*_{\mathcal{E}}(p) - \delta_{\mathbb{R}^\ell_+}(p)\} \geq \sup_e \inf_p \; S_{\mathcal{E}}(p, e) = \sup_e \{\nu_{\mathcal{E}}(e) - \delta^*_{\mathbb{R}^\ell_+}(e)\}$$

The saddle function is convex in p and concave in e.

Proposition 4.1.1 *There exists saddle points* $(p^W, \mathbf{0})$

$$S_{\mathcal{E}}(p^W, e) \leq S_{\mathcal{E}}(p^W, \mathbf{0}) = \nu_{\mathcal{E}}(\mathbf{0}) - p^W \cdot \mathbf{0} = \nu^*_{\mathcal{E}}(p^W) \leq S_{\mathcal{E}}(p, \mathbf{0}), \quad \forall p, e \in \mathbb{R}^\ell,$$

and $\langle e^W_i \rangle$, *such that* $(p^W, \langle e^W_i \rangle)$ *is a WE with*

$$e^W_i \in \partial \nu^*_i(p^W), \forall i \quad \sum_i \mathfrak{e}^W_i = \mathbf{0}$$

The set of saddle point prices p^W *is convex.*

Proof: It follows from the fundamental theorem of linear programing.

4.1.2 non-QL WE

Without the money commodity, a price normalization for exchange is

$$P^\ell = \{p = \langle p_c \rangle \in \mathbb{R}^\ell : p_c \geq 0, \sum_c p_c = 1\}$$

with $\delta_{P^\ell}(p) \in \{0, -\infty\}$, the concave indicator function of P^ℓ, and conjugate

$$\delta^*_{P^\ell}(e) = \inf_p \{p \cdot e - \delta_{P^\ell}(p)\}$$

Aggregate utility and indirect utility for the non-QL model are

$$\nu_{\mathcal{E}}(e \,|\, p) = \sup\{\sum_i \nu_{\mathcal{E}_i}(e_i \,|\, p) : \sum_i e_i = e\} \quad \nu^*_{\mathcal{E}}(p; 0) = \sum_i \nu^*_i(p; 0)\}$$

The saddle function is

$$S_{\mathcal{E}}(p, e; 0) = \nu_{\mathcal{E}}(e \,|\, p) - p \cdot e - \delta_{P^\ell}(p)$$

with minimax inequality

$$\inf_p \sup_e S_{\mathcal{G}}(p, e; 0) = \inf_p \{\nu^*_{\mathcal{E}}(p; 0) - \delta_{P^\ell}(p)\} \geq \sup_e \inf_p \; S_{\mathcal{E}}(p, e; 0) = \sup_e \{\nu_{\mathcal{E}}(e \,|\, p) - \delta^*_{P^\ell}(e)\}$$

This saddle function does not exhibit conjugate duality: $\nu^*_i(p; 0)$ is only quasi-convex and their sum, $\nu^*_{\mathcal{E}}(p; 0)$, need not be even quasi-convex.

Following Nikaido [1960], the non-QL model can be converted into a convex family of QL models. Letting $\mathbf{A} = \{\mathbf{a} = (\alpha_1, \ldots, \alpha_n) : \alpha_i \geq 0, \sum_i \alpha_i = 1\}$, $\mathcal{E}(\mathbf{a}) = \langle \alpha_i \nu_i \rangle$ defines an exchange economy with QL utilities.

Proposition 4.1.2 *Saddle points* $(p^W, \mathbf{0})$

$$S_{\mathcal{E}(\mathbf{a})}(p^W, e) \leq S_{\mathcal{E}(\mathbf{a})}(p^W, \mathbf{0}) = \nu^*_{\mathcal{E}(\mathbf{a})}(p^W) = \nu_{\mathcal{E}(\mathbf{a})}(\mathbf{0}) - p^W \cdot \mathbf{0} \leq S_{\mathcal{E}(\mathbf{a})}(p^W, \mathbf{0}), \quad \forall p, e \in \mathbb{R}^\ell$$

exist. Moreover, there exists $\mathbf{a}$ *with* $\alpha_i > 0$ *such that* $\langle e^W_i \rangle$, *such that* $(p^W, \langle e^W_i \rangle)$ *is a WE with*

$$e^W_i \in \partial[\alpha_i \nu_i]^*(p^W; 0), \forall i \quad \sum_i \mathfrak{e}^W_i = \mathbf{0}$$

If p^W_1, p^W_2 *are saddle points, their convex combination may not be a saddle-point.*

Proof: See Appendix.

Remark 7: When there is more than one fixed-point, their minimum values need not be the same, i.e,

$$\nu^*_{\mathcal{E}}(p^W_1; 0) = \nu_{\mathcal{E}}(\mathbf{0}; p^W_1) \neq \nu^*_{\mathcal{E}}(p^W_2; 0) = \nu_{\mathcal{E}}(\mathbf{0}; p^W_2).$$

Therefore, while each non-QL WE satisfies what may be regarded as a "local minimax condition," global comparisons are not meaningful. Hence, even though utilities are concave, ordinal interpretations of utility are a consequence of the non-QL budget constraint.

4.2 Games

4.2.1 Hannan Equilibrium

There is an HE for $\langle \mathfrak{E}_i \rangle$ and another for $\langle \mathfrak{E}_i^0 \rangle$.

The concave indicator of P is $\delta_P(p) \in \{0, -\infty\}$ with conjugate

$$\delta_P^*(\mathfrak{e}) = \inf_p \{p \cdot \mathfrak{e} - \delta_P(p).\}$$

The analogs for $\langle \mathfrak{E}_i^0 \rangle$ of the aggregate utility $\nu_{\mathcal{E}}(e)$ and conjugate functions $\nu_{\mathcal{E}}^*(p)$ for exchange are

$$\delta_{\mathfrak{E}^0}(\mathfrak{e}) = \inf \{\sum_i \delta_{\mathfrak{E}_i^0}(\mathfrak{e}_i) : \sum_i \mathfrak{e}_i = \mathfrak{e}\} \quad \delta_{\mathfrak{E}^0}^*(p) = \sum_i \delta_{\mathfrak{E}_i^0}(p)$$

The saddle function is

$$S_{\mathcal{G}^0}(p, \mathfrak{e}) = p \cdot \mathfrak{e} - \delta_{\mathfrak{E}^0}(\mathfrak{e}) - \delta_P(p)$$

with minimax inequality

$$\inf_p \sup_{\mathfrak{e}} S_{\mathcal{G}^0}(p, \mathfrak{e}) = \inf_p \{\delta_{\mathfrak{E}^0}^*(p) - \delta_P(p)\} \geq \sup_{\mathfrak{e}} \inf_p S_{\mathcal{G}^0}(p, \mathfrak{e}) = \sup_{\mathfrak{e}} \{\delta_P^*(\mathfrak{e}) - \delta_{\mathfrak{E}^0}(\mathfrak{e})\}$$

The saddle function is convex in p and concave in $\mathfrak{e}$.

Proposition 4.2.1 *Saddle points* $(p^H, \mathbf{0})$

$$S_{\mathcal{G}^0}(p^H, \mathfrak{e}) \leq S_{\mathcal{G}^0}(p^H, \mathbf{0}) = p^H \cdot \mathbf{0} - \delta_{\mathfrak{E}}(\mathbf{0}) = \delta_{\mathfrak{E}}^*(p^H) \leq S_{\mathcal{G}^0}(p, \mathbf{0}), \quad \forall p, \mathfrak{e} \in \mathbb{R}^A$$

exist. Moreover, the set of saddle points is convex and

$$\mathbf{0} \in \partial \delta_{\mathfrak{E}_i^0}(p^H), \forall i$$

Aggregate convexity conditions for $\delta_{\mathfrak{E}}(\mathfrak{e})$ and $\delta_{\mathfrak{E}}^*(p)$ apply to $\langle \mathfrak{E}_i \rangle$ and the same methods of proof imply a saddle-point $(p^H, \mathfrak{e}^H)$. However.

$$S_{\mathcal{G}}(p^H, \mathfrak{e}) \leq S_{\mathcal{G}}(p^H, \mathfrak{e}^H) = p^H \cdot \mathfrak{e}^H - \delta_{\mathfrak{E}}(\mathfrak{e}^H) = \delta_{\mathfrak{E}}^*(p^H) \leq S_{\mathcal{G}}(p, \mathfrak{e}^H), \quad \forall p, \mathfrak{e} \in \mathbb{R}^A$$

where $p^H \cdot \mathfrak{e}^H \leq 0$.

4.2.2 Correlated Equilibrium: Delegated choice with unrestricted prices

With delegate choice, the analogs of $\delta_{\mathfrak{E}^0}(\mathfrak{e})$ and $\delta_{\mathfrak{E}^0}^*(p)$ are

$$\delta_{\mathfrak{E}_A}(\mathfrak{e}) = \inf \{\sum_i \sum_{a_i} \delta_{\mathfrak{E}_{a_i}}(\mathfrak{e}_{a_i}) : \sum_i \sum_{a_i} \mathfrak{e}_{a_i} = \mathfrak{e}\} \quad \delta_{\mathfrak{E}_A}^*(p) = \sum_i \sum_{a_i} \delta_{\mathfrak{E}_{a_i}}^*(p)$$

The saddle function is

$$S_{\mathcal{G}_A}(p, \mathfrak{e}) = p \cdot \mathfrak{e} - \delta_{\mathfrak{E}_A}(\mathfrak{e}) - \delta_P(p)$$

with minimax inequality

$$\inf_p \sup_{\mathfrak{e}} S_{\mathcal{G}_{\mathcal{A}}}(p,\mathfrak{e}) = \inf_p \{\delta^*_{\mathfrak{E}_A}(p) - \delta_P(p)\} \geq \sup_{\mathfrak{e}} \inf_p S_{\mathcal{G}_{\mathcal{A}}}(p,\mathfrak{e}) = \sup_{\mathfrak{e}} \{\delta^*_P(\mathfrak{e}) - \delta_{\mathfrak{E}_A}(\mathfrak{e})\}$$

The saddle function is convex in p and concave in $\mathfrak{e}$.

Proposition 4.2.2 *Saddle points* $(p^C, \mathbf{0})$

$$S_{\mathcal{G}_{\mathcal{A}}}(p^C,\mathfrak{e}) \leq S_{\mathcal{G}_{\mathcal{A}}}(p^C,\mathbf{0}) = p^C \cdot \mathbf{0} - \delta_{\mathfrak{E}_A}(\mathbf{0}) = \delta^*_{\mathfrak{E}_A}(p^) \leq S_{\mathcal{G}_{\mathcal{A}}}(p,\mathbf{0}), \quad \forall p, \mathfrak{e} \in \mathbb{R}^A$$

exist. Moreover, the set of saddle-points is convex and,

$$\mathfrak{e}^C_{a_i} = \mathbf{0} \in \partial\delta^*_{\mathfrak{E}_{a_i}}(p^C) \quad \forall a_i, p^C(a_i) > 0$$

Remark 8: (PRIOR SADDLE-POINT PROOFS OF CE) Hart and Schmeidler [1989] also give a saddle-point proof of CE. They formulate the problem as a *two-person, zero sum* game with a player controlling prices wishing to minimize and an aggregate other wishing to maximize. As a zero-sum, the payoffs to the two are always the same. The formulation above can also be said to have two players: a price minimizer and the payoffs resulting from the sum of the individuals' deviation gain maximizing responses. In this case, however, the sum of the payoffs varies with prices and need only be zero-sum at equilibrium. Nau and MacCardle [1990] exploit duality formulations from linear programming. Myerson [1997] exploits complementary slackness conditions in linear programming as they explicitly relate to games.

4.2.3 Nash Equilbrium: Non-delegated choice with restricted prices

Define the indicator of $\mathfrak{T}[Q]$ is $\delta_{\mathfrak{T}[Q]}(p) \in \{0, -\infty\}$, i.e., $\delta_{\mathfrak{T}[Q]}(p)) = 0$ implies $p \in \mathfrak{T}[Q]$. Note the set on which $\delta_{\mathfrak{T}[Q]}(p) = 0$ is not convex. Its conjugate

$$\delta^*_{\mathfrak{T}[Q]}(\mathfrak{e}) = \inf\{p \cdot \mathfrak{e} - \delta_{\mathfrak{T}[Q]}(p)\}$$

The aggregate utility and aggregate conjugate functions are

$$\delta_{\mathfrak{E}}(\mathfrak{e}) = \inf \Big\{ \sum_i \delta_{\mathfrak{E}_i}(\mathfrak{e}_i) : \sum_i \mathfrak{e}_i = \mathfrak{e} \Big\} \quad \delta^*_{\mathfrak{E}}(p) = \sum_i \delta^*_{\mathfrak{E}_i}(p).$$

The saddle function is

$$S_{\mathcal{G}}(p,\mathfrak{e};\mathfrak{T}) = p \cdot \mathfrak{e} - \delta_{\mathfrak{E}}(\mathfrak{e}) - \delta_{\mathfrak{T}[Q]}(p)$$

with minimax inequality

$$\inf_p \sup_{\mathfrak{e}} S_{\mathcal{G}}(p,\mathfrak{e};\mathfrak{T}) = \inf_p \{\delta^*_{\mathfrak{E}}(p) - \delta_{\mathfrak{T}[Q]}(p)\} = \inf_q \ \delta^*_{\mathfrak{E}}(p) \geq \sup_{\mathfrak{e}} \inf_p S_{\mathcal{G}}(p,\mathfrak{e};\mathfrak{T}) = \sup_{\mathfrak{e}} \{\delta^*_{\mathfrak{T}[Q]}(\mathfrak{e}) - \delta_{\mathfrak{E}}(\mathfrak{e})\}$$

The saddle function is concave in $\mathfrak{e}$, but it is *not* convex in p because $\mathfrak{T}[Q]$ is not convex in P. Nevertheless, just as non-convexity of the saddle function in p for non-QL WE does not preclude a saddle-point,

Proposition 4.2.3 *A saddle-point* $(\mathfrak{T}(q^N), \mathfrak{e}^N)$

$$S_{\mathcal{G}}(\mathfrak{T}(q^N), \mathfrak{e}; \mathfrak{T}) \leq S_{\mathcal{G}}(\mathfrak{T}(q^N), \mathfrak{e}^N; \mathfrak{T}) = p \cdot \mathfrak{e}^N - \delta_{\mathfrak{E}}(\mathfrak{e}^N) = \delta^*_{\mathfrak{E}}(\mathfrak{T}(q^N)) \leq S_{\mathcal{G}}(p, \mathfrak{e}^N; \mathfrak{T}) \quad \forall p, \mathfrak{e} \in \mathbb{R}^A$$

exists. Moreover, there exists $\langle \mathfrak{e}_i^N(q_i^N) \rangle$,

$$\mathfrak{e}_i^N(q_i^N) \in \partial \delta^*_{\mathfrak{E}_i}(\mathfrak{T}(q^N)), \quad \mathfrak{T}(p^N) \cdot \mathfrak{e}_i^N(p_i^N) = 0, \forall i$$

The set of saddle points is not convex.

Proof: See Appendix.

NE is a HE in which $\mathfrak{E}_i^0$ can be replaced by $\mathfrak{E}_i$ because prices are restricted to $\mathfrak{T}[Q]$. The restriction implies that NE is a "self-saddle-point," i.e., $\mathfrak{e}_i^N(q_i^N) \in \partial \delta^*_{\mathfrak{E}_i}(\mathfrak{T}(q^N)$.

5 Equilibrium in $\mathfrak{T}[Q]$ without a fixed point

5.1 Identical Interest Games: "Complete-in-itself Duality" for $\mathfrak{T}[Q]$

Interdependence in a strategic form game is defined by utilities $\mathfrak{u}_i(a_1, \ldots, a_n)$ with the understanding that i controls the choices $a_i \in A_i$. If i controlled A_j and *vice-versa*, that would be a different game. Or, the name i could be given to j, and j to i, so that individuals' names are defined by the choices they control. In other words, a game in strategic form is defined by individual utility functions on A and the convention that i controls A_i. Monderer and Shapley [1996a,1996b] introduced *identical interest games* having the property that the assignment of names to choices does not matter, assuming each individual controls exactly one A_i. Namely, when there exists $\mathfrak{u}$ such that in terms of deviation gains it is as if $\mathfrak{u}_i = \mathfrak{u}$ for all i.

$$\mathfrak{e}_{\mathbf{I}}(a)[b_j] = \mathfrak{u}(a)[b_j] - \mathfrak{u}(a)[b_j] = \mathfrak{e}_i(a)[b_j] = \mathfrak{u}_i(b_j, a_{-j}) - \mathfrak{u}_i(a_j, a_{-j}), \quad \forall i, j \tag{5.1.1}$$

Denote the convex polyhedron of aggregate deviation gains by $\mathfrak{E}_{\mathbf{I}}$.

Monderer and Shapley [1996b] show that such games exhibit a *potential function.* In the formulations used here, they show that bounding hyperplanes identifying $\mathfrak{E}_{\mathbf{I}}$ are contained in $\mathfrak{T}[Q]$. I.e., if $\mathfrak{E}_{\mathbf{I}}$ is obtained from $\mathfrak{u}$ and $\partial \delta^*_{\mathfrak{E}_{\mathbf{I}}}(\mathfrak{T}(q))$ are the maximizing choices from $\mathfrak{T}[Q]$, and $\mathfrak{E}$ is another deviation gain polyhedron with

$$\partial \delta^*_{\mathfrak{E}}(\mathfrak{T}(q)) = \partial \delta^*_{\mathfrak{E}_{\mathbf{I}}}(\mathfrak{T}(q)) \tag{5.1.2}$$

then $\mathfrak{E}$ is a deviation gains polyhedron obtained from a utility function $\mathfrak{u} + \alpha$.

In addition, M&S showed that for $\mathfrak{E}_{\mathbf{I}}$ there exist pure strategy NE. I.e., $\mathfrak{e}^a := \mathfrak{e}(\mathbf{1}_a)$ such that

$$\delta^*_{\mathfrak{E}_{\mathbf{I}}}(\mathfrak{T}(q)) + \delta_{\mathfrak{E}_{\mathbf{I}}}(\mathfrak{e}^a) = \mathfrak{T}(q) \cdot \mathfrak{e}^a = \delta^*_{\mathfrak{E}_{\mathbf{I}}}(\mathbf{1}_a) + \delta_{\mathfrak{E}_{\mathbf{I}}}(\mathfrak{e}^a) = 0 \tag{5.1.3}$$

Remarkably, the identification of $\mathfrak{E}_{\mathbf{I}}$ from $\mathfrak{T}[Q]$ does not rule out the possibility that $\mathfrak{T}[Q]$ does not exhaust the exposed faces of $\mathfrak{E}_{\mathbf{I}}$. I.e., there can exist $p \in P \backslash \mathfrak{T}[Q]$ such that

$$\delta^*_{\mathfrak{E}_{\mathbf{I}}}(p) + \delta_{\mathfrak{E}_{\mathbf{I}}}(\mathfrak{e}) = p \cdot \mathfrak{e} \neq \delta^*_{\mathfrak{E}_{\mathbf{I}}}(\mathfrak{T}(q)) + \delta_{\mathfrak{E}_{\mathbf{I}}}(\mathfrak{e}), \forall q \in Q \tag{5.1.4}$$

Example 1 in Section 3.3.3 is an identical interest game with an HE yielding negative deviation gains.

5.2 Two-person Zero-sum Games

Equilibrium in $\mathfrak{T}[\Delta(A_1), \Delta(A_2)]$ for 2-0 games was the launching pad for NE. vNM's original proof by fixed-point methods was given a more elementary proof that did not apply to Nash's extension. It is of some interest to show that a starting point for equilibrium with respect to 2-0 games can be formulated as a pricing problem amenable to elementary methods that is applicable to more general games. It consists of three players: a price-setter choosing among $P = \{p(a_1, a_2) : (a_1, a_2) \in A_1 \times A_2), p(a_1, a_2) \geq 0, \sum_{a_1, a_2} p(a_1, a_2) = 1\}$ whose objective is to minimize the gains from 1 and 2, whose objectives are to maximize.

The deviation gain for i can be written as:

$$p \cdot \mathfrak{e}_i(\mathfrak{z}_i) = \sum_a p(a) \sum_{b_i} [\mathfrak{u}_i(b_i, a_{-i}) - \mathfrak{u}_i(a_i, a_{-i})] \mathfrak{z}_i(b_i) \tag{5.2.1}$$

$$= \sum_{a_{-i}} \sum_{b_i} \left[p_{-i}(a_{-i}) \mathfrak{z}_i(b_i)\right] \mathfrak{u}_i(b_i, a_{-i}) - \sum_a p(a) \mathfrak{u}_i(a) = (p_{-i} \otimes \mathfrak{z}_i) \cdot \mathfrak{u}_i - p \cdot \mathfrak{u}_i \tag{5.2.2}$$

The pricing problem of the price-setter is

$$\min_p \max_{\mathfrak{e}_1 \in \mathfrak{E}_1, \mathfrak{e}_2 \in \mathfrak{E}_2} p \cdot (\mathfrak{e}_1 + \mathfrak{e}_2) \tag{5.2.3}$$

$$= \min_p \max_{\mathfrak{z}_1 \in \Delta(A_1), \mathfrak{z}_2 \in \Delta(A_2)} [(p_2 \otimes \mathfrak{z}_1) \cdot \mathfrak{u}_1 - p \cdot \mathfrak{u}_1] + [(p_1 \otimes \mathfrak{z}_2) \cdot \mathfrak{u}_2 - p \cdot \mathfrak{u}_2] \tag{5.2.4}$$

$$= \min_p \max_{\mathfrak{z}_1 \in \Delta(A_1), \mathfrak{z}_2 \in \Delta(A_2)} [(p_2 \otimes \mathfrak{z}_1) \cdot \mathfrak{u}_1 + (p_1 \otimes \mathfrak{z}_2) \cdot \mathfrak{u}_2] \text{ since } \mathfrak{u}_1 + \mathfrak{u}_2 = \mathbf{0} \text{ (0-sum)} \tag{5.2.5}$$

The price-setter is not interested in possible correlation in p anymore, but only interested in choosing the marginal distributions p_1 and p_2 to minimize the aggregate deviation gain.

By abusing notations for $\mathfrak{u}_i$s to be matrices of dimension $|A_1| \times |A_2|$ (following the classical 2-0 game formulation), the pricing problem is additively decomposed into two minimax problems.

$$\min_p \left[\max_{\mathfrak{z}_1} (p_2 \otimes \mathfrak{z}_1) \cdot \mathfrak{u}_1 + \max_{\mathfrak{z}_2} (p_1 \otimes \mathfrak{z}_2) \cdot \mathfrak{u}_2\right] = \min_p \left[\max_{\mathfrak{z}_1} \mathfrak{z}_1 \mathfrak{u}_1 {p_2}^T + \max_{\mathfrak{z}_2} p_1 \mathfrak{u}_2 {\mathfrak{z}_2}^T\right] \tag{5.2.6}$$

$$= \min_{p_2} \max_{\mathfrak{z}_1} \mathfrak{z}_1 \mathfrak{u}_1 {p_2}^T + \min_{p_1} \max_{\mathfrak{z}_2} p_1 \mathfrak{u}_2 {\mathfrak{z}_2}^T \tag{5.2.7}$$

Given $\mathfrak{u}_2 = -\mathfrak{u}_1$, the second minimax problem becomes

$$\min_{p_1} \max_{\mathfrak{z}_2} p_1 \mathfrak{u}_2 {\mathfrak{z}_2}^T = \min_{p_1} \max_{\mathfrak{z}_2} p_1(-\mathfrak{u}_1){\mathfrak{z}_2}^T \tag{5.2.8}$$

$$= -\max_{p_1} \min_{\mathfrak{z}_2} p_1 \mathfrak{u}_1 {\mathfrak{z}_2}^T \tag{5.2.9}$$

$$= -\min_{\mathfrak{z}_2} \max_{p_1} p_1 \mathfrak{u}_1 {\mathfrak{z}_2}^T \tag{5.2.10}$$

Replacing notation $\mathfrak{z}_2$ with p_2, and p_1 with $\mathfrak{z}_1$, the second minimax problem is shown to be identical to the negative of the first problem.

Thus, we find that (i) the aggregate deviation gain is zero, and (ii) the solution of the pricing problem is the same as the classical minimax solution of 2-0 games in which $p_1 = \mathfrak{z}_1$ and $p_2 = \mathfrak{z}_2$.

6 Convergence of Excess Demands in Exchange and Games

In exchange equilibrium is found by adjusting prices to aggregate excess demands. That method can also be applied to games since equilibration of excess demands also characterizes non-cooperative equilibrium.

The standard description of price adjustment assumes individual excess demands are single-valued and continuous. The polyhedral properties of conjugate functions underlying excess demands for exchange and games implies they are multi-valued and discontinuous. Hence, convergence of *prices* must hold for *any* selection of individual excess demands, an evidently more demanding requirement. It fullfillment is associated with a qualification that only the time average of aggregate excess demands, in exchange and games, converge to $\mathbf{0}$. Time average of quantities is a central feature of price-taking play methods of convergence in Part II.

Differences in the methods of proving existence correspond to differences with respect to convergence of price-taking excess demands. Where conjugate duality suffices for existence, as in QL WE for exchange, and HE and CE in games, it also suffices for convergence. In non-QL WE, where fixed-point arguments are required, it is well-known that convergence need not obtain. Similarly, the excess demand approach to convergence for NE also fails.

6.1 Exchange

6.1.1 QL WE

Aggregate excess demands in $\mathcal{E}$ are given by

$$e(p) = \sum_i e_i(p) \in \sum_i \partial \nu_i^*(p) := \partial \nu_E^*(p) \subset \mathbb{R}^\ell$$

an arbitrary selection from $\partial \nu_E^*(p)$. To accommodate discontinuities, i.e., jumps, in $e(p)$, price adjustment is modeled as less responsive over time to current excess demands. Instead of $p' = p + se(p)$, $s > 0$,

$$p^{t+1} = p^t + t^{-1}e(p^t)$$

To conform to the restriction that p^{t+1} must lie in $\mathbb{R}^\ell_+$, let

$$\mathrm{Proj}_{\mathbb{R}^\ell_+}[p + t^{-1}\mathfrak{e}(p)] = \operatorname*{argmin}_{p'}\{\delta_{\mathbb{R}^\ell_+}(p') + 2^{-1}|p' - [p + t^{-1}\mathfrak{e}(p)]|^2\}$$

Hence, $p^t + t^{-1}e(p^t) \in \mathbb{R}^\ell_+$ implies $\mathrm{Proj}_{\mathbb{R}^\ell_+}[p^t + t^{-1}e(p^t)] = p^t + t^{-1}e(p^t)$; otherwise, if $p^t_c + t^{-1}e_c(p^t) < 0, c = 1, \ldots, \ell$, $p_c = 0$. (Note again that the price adjustment allows for discontinuous response $e(p^t)$ unlike a smooth version of economy, e.g., Uzawa [1960])

Price adjustment is

$$p^{t+1} = \mathrm{Proj}_{\mathbb{R}^\ell_+}[p^t + t^{-1}\mathfrak{e}(p^t)]$$

The standard for quantity convergence is weakened to averaging

$$\overline{e}^t = t^{-1}\big(\sum_{\tau=0}^{t-1} e^\tau\big)$$

with $\overline{e}^{t+1}$ determined by the sequence $\{p^1, p^2, \ldots, p^t\}$ and the choices $e(p^\tau), \tau = 1, 2, \ldots, t$ and

$$\overline{e}^{t+1} = \overline{e}^t + \tfrac{1}{t+1}[e(p^{t+1}) - \overline{e}^t]$$

Proposition 6.1.1 *Price adjustment in $\mathcal{E}$ defined by $p^{t+1} = Proj_{\mathbb{R}^\ell_+}[p^t - t^{-1}\mathfrak{e}(p^t)]$ implies*

$$p^t \to p^W \text{ and } \overline{e}^t \to \mathbf{0}$$

Proof: The convergence of p^{t+1} to a point is established by Shor [1985, Theorem 2.2], as refined by Anstreicher and Wolsey [2009, Theorem 3].

Example 2: (Convergence of average excess demands)
There is a single individual; hence the subscript i is omitted in the following, and a single (non-money) commodity. Let $E^F = \{-1, 0, 1\}$ and $\nu(e) = e$, $e \in E$. (Again, $\mathbf{0} = 0$.) It is readily established that

$$-\nu^*(p) = \max\{1 - p, p - 1\}.$$

Equilibrium price and quantity are $(p^0 = 1, e^0 = 0)$, while

$$\partial\nu^*(p) = \begin{cases} \{-1, 0, 1\} & \text{if } p = 1, \\ -1 & \text{if } p > 1, \\ 1 & \text{if } p < 1. \end{cases}$$

Failure of demand to be differentiable implies that $e(p)$ is discontinuous at $p = 1$. For $p^1 = 3$, prices intially fall, going below below 1 and then rising as $p^t \to 1$, while $e^t(p^t)$ does not converge, oscillating between -1 and 1; however, the average value of excess demands converges to 0.

Remark 9: Compared to the gradient algorithm for $\mathcal{E}$, the subgradient algorithm for $\widehat{\mathcal{E}}_F$ does not imply steadily decreasing aggregate utility. Moreover, the convergence requirement for quantities is weakened to $\overline{e}^t \to \mathbf{0}$, with the average positive excess demands balanced by their average negative excess demands over time, allowing $\|e(p^t)\|$ to be bounded away from $\mathbf{0}$.

Instead of finite available activities, we could assume continuum activities with differentiable utility functions, call this economy $\mathcal{E}^{\nabla}$. Price adjustment in $\mathcal{E}^{\nabla}$ is achieved via a gradient algorithm relying on the single-valued and continuity properties of $\nabla \nu_i^*(p)$.

Aggregate excess demand at p is

$$e(p) = \sum_i e_i(p) = \sum_i \nabla \nu_i^*(p) = \nabla \mathcal{V}_{\mathcal{E}}^*(p)$$

Prices adjustment to excess demand according to the 'Law of Supply and Demand' as

$$p^{t+1} = p^t - s[-\nabla \mathcal{V}_{\mathcal{E}}^*(p^t)] = p^t + se(p^t), \tag{6.1.1}$$

where s is the rate at which price changes respond to excess demands. This description, known as Walrasian *tâtonnement*, is a gradient version of the primal-dual algorithm in LP in which (feasible) adjustments in the dual (prices) are paired with their infeasible primal counterparts (non-zero excess demands) until prices are found such that feasibility, $e(p^0) = \mathbf{0}$, is achieved.

Proposition 6.1.2 *Price adjustment in $\mathcal{E}^{\nabla}$ defined by (6.1.1) implies*

$$p^t \to p^0 \text{ and } e(p^t) = \nabla \mathcal{V}_{\mathcal{E}}^*(p^t) \to e(p^0) = \nabla \mathcal{V}_{\mathcal{E}}^*(p^0) = \mathbf{0}.$$

The gradient algorithm is always descending towards its goal, i.e., $-\mathcal{V}_{\mathcal{E}}^*(p^{t+1}) < -\mathcal{V}_{\mathcal{E}}^*(p^t)$. Moreover, $\mathcal{V}_{\mathcal{E}}^*(p^t) \to \mathcal{V}_{\mathcal{E}}^*(p^0)$ faster than $\|p^t - p^0\| \to 0$.

6.1.2 non-QL WE

Polyhedral concavity does not eliminate income effects of price changes from non-QL budget constraints (See counterexamples in Scarf [1960]). *Weak gross substitutability* holds for QL models, but not for non-QL models. If aggregate demands in non-QL model mimic certain properties of QL model, convergence may occur.

6.2 Games

Price adjustment follows the pattern for WE with prices adjusting to the sum of excess demands.

6.2.1 Hannan Equilibrium

Aggregate excess demands is an arbitrary selection

$$\mathfrak{e}^0(p) = \sum_i \mathfrak{e}_i^0(p) \in \sum_i \partial\delta^*_{\mathfrak{E}_i^0}(p) := \partial\delta^*_{\mathfrak{E}^0}(p) \subset \mathbb{R}^A$$

Price adjustment exhibits declining weights over time,

$$p^t - t^{-1}\mathfrak{e}^0(p^t);$$

and conforms to the restriction the adjustment must lie in P via

$$\text{Proj}_P[p - t^{-1}\mathfrak{e}^0(p)] = \underset{p'}{\text{argmin}}\,\{\delta(p') + 2^{-1}|p' - [p - t^{-1}\mathfrak{e}^0(p)]|^2\}$$

$$\text{Proj}_P[p - t^{-1}\mathfrak{e}^0(p)] = \underset{p'}{\text{argmin}}\,\{\delta(p') + 2^{-1}|p' - [p - t^{-1}\mathfrak{e}^0(p)]|^2\}$$

Letting $\bar{\mathfrak{e}}^{t0} = t^{-1}(\sum_{\tau=0}^{t-1}(\mathfrak{e}^0)^t)$, the sequence of averages is

$$\bar{\mathfrak{e}}^{(t+1)0} = \bar{\mathfrak{e}}^{t0} + \tfrac{1}{t+1}[\mathfrak{e}^0(p^t) - \bar{\mathfrak{e}}^{t0}]$$

Proposition 6.2.1 *Price adjustment for HE defined by $p^{t+1} = Proj_P[p^t - t^{-1}\mathfrak{e}^0(p^t)]$ implies*

$$p^t \to p^H \;\; \textit{and} \;\; \bar{\mathfrak{e}}^{t0}(p^t) \to \mathbf{0}.$$

6.2.2 Correlated Equilibirum

Aggregate excess demands is an arbitrary selection

$$\mathfrak{e}_A(p) = \sum_i \sum_{a_i} \mathfrak{e}_{a_i}(p) \in \sum_i \sum_{a_i} \partial\delta^*_{\mathfrak{E}_{a_i}}(p) = \partial\delta^*_{\mathfrak{E}_A}(p) \in \mathbb{R}^A$$

Again, price adjustment is modified to exhibit declining weights over time,

$$p^t - t^{-1}\mathfrak{e}_A(p^t);$$

and to conform to the restriction the adjustment must lie in P via

$$\text{Proj}_P[p - t^{-1}\mathfrak{e}_A(p)] = \underset{p'}{\text{argmin}}\,\{\delta(p') + 2^{-1}|p' - [p - t^{-1}\mathfrak{e}_A(p)]|^2\}$$

Projection is continuously differentiable – smooths the jumps in $\mathfrak{e}_{a_i}(p)$. Yet, another kind of smoothing is required, i.e., the adjustment speed should decrease at the rate of $1/t$. In general, adjustment speed s^t needs to satisfy $s^t \to 0$, $\lim_{t\to\infty}\sum_t s_t = \infty$ and $\lim_{t\to\infty}\sum_t s_t^2 < \infty$ for point convergence, i.e., it should decrease slow enough ($\lim_{t\to\infty}\sum_t s_t = \infty$), but not too slow ($\lim_{t\to\infty}\sum_t s_t^2 < \infty$). (See Anstreicher and Wolsey [2009].)

The standard for convergence of excess demands is weakened to averaging. Letting $\bar{\mathfrak{e}}_A^t = t^{-1}(\sum_{\tau=0}^{t-1}\mathfrak{e} + A^t)$, the sequence of averages is

$$\bar{\mathfrak{e}}_A^{t+1} = \bar{\mathfrak{e}}_A^t + \tfrac{1}{t+1}[\mathfrak{e}_A(p^t) - \bar{\mathfrak{e}}_A^t]$$

Proposition 6.2.2 *Price adjustment for CE defined by* $p^{t+1} = Proj_P[p^t - t^{-1}\mathfrak{e}_A(p^t)]$ *implies*

$$p^t \to p^C \text{ and } \bar{\mathfrak{e}}_A(p^t) \to \mathbf{0}.$$

Example 3: (Convergence of average excess demands) In the two-person, zero-sum 'matching-pennies' game with $A = \{(H,H),(T,T),(H,T),(T,H)\}$ and $\mathfrak{u}_1(H,H) = \mathfrak{u}_1(T,T) = 1$ and $\mathfrak{u}_1(H,T) = \mathfrak{u}_1(T,H) = -1$, the only equilibrium is $p^0 = (1/4, 1/4, 1/4, 1/4)$. Therefore, $p^t \to p^0$. As in Example 1, it is readily verified that whenever $p^t \neq p^0$, $(\mathfrak{e}_1(\mathfrak{z}_1(p^t)), \mathfrak{e}_2(\mathfrak{z}_2(p^t)) \in \{(-2,2,-2,2),(2,-2,2,-2)\}$ is bounded away from $\mathbf{0}$; however, average excess demands converge to $\mathbf{0}$.

6.2.3 Nash Equilibrium

Consider a similar price adjustment for Nash equilibrium based on the following subgradient:

$$\mathfrak{e}(\mathfrak{T}(q)) \in \sum_i \partial\delta^*_{\mathfrak{E}_i}(\mathfrak{T}(q)) := \partial\delta_{\mathfrak{E}}(\mathfrak{T}(q))$$

To conform to the restriction for NE, the adjustment must lie in $\mathfrak{T}[Q]$. However, the problem is that $\mathfrak{T}[Q]$ is not convex in P, so projection is not convex anymore. A significant modification of the price adjustment would be required for Nash equilibrium.

7 Concluding Remarks

The relation between exchange and games, above, is based on demonstrations of the similarities between *sums* of vectors of individual commodity excess demands in exchange and *sums* of vectors of individual deviation gains in games, all of which originate from the choice of activity vectors, $z_i \in \Delta(A_i)$ in exchange, or $z_i, z_{a_i} \in \Delta(A_i)$ in games.

There is formulation of games, with no counterpart in exchange, in which the *product* of activity choices, either its Cartesian product for NE or its tensor product for HE and CE, is the focus of attention. These formulations are the setting for convergence to non-cooperative equilibrium as modeled by game theorists. The earliest versions, called *fictitious play*, concerned adjustments of $\times_i z_i \in Q$ for NE. (Brown [1951], Robinson [1951], Shapley [1964]). Later extensions for HE and CE are based on $\otimes_i z_i$. (Hannan [1957], Fudenberg and Levine [1998], Foster and Vohra [1997], Hart and Mas-Colell [2000]). In each of these instances, the choice of z_i is, or can be given, the same pricing rationale underlying the determination of deviation gains employed above. Demonstrations of that claim is the subject of Ostroy and Song [2023].

8 Appendix: Proofs

8.1 Proof for Proposition 4.1.3

The proof, following Nikaido (1960), proceed as in the diagram below: (i) start with utility functions $\langle \nu_i \rangle$, (ii) find $\{p, \langle e_i \rangle\}$ as a fixed point of mapping $F_{\mathcal{E}}$ defined below, (iii) find proper weight $\mathbf{a} := \langle \bar{\alpha}_i \rangle$ connecting up the optimization with QL and non-QL utilities, (iv) formulate value function $\nu_{\mathcal{E}(\overline{\alpha})}$ using the weight, then (v) describe a saddle function to characterize an equilibrium as a saddle point.

$$\langle \nu_i \rangle \xrightarrow{\text{fixed point of } F_{\mathcal{E}}} \{p, \langle e_i \rangle\} \xrightarrow{[\bar{\alpha}_i \nu_i]^*(p^W) = \bar{\alpha}_i \nu_i^*(p^W;0)} \langle \overline{\alpha}_i \rangle \to \nu_{\mathcal{E}(\overline{\alpha})} \to \text{saddle function}$$

Consider correspondence $F_{\mathcal{E}} : P^\ell \rightrightarrows P^\ell$:

$$F_{\mathcal{E}}(p) \ni \frac{p + [\sum_i e_i(p)]^+}{1 + \sum_c [\sum_i e_i(p)]^+(c)}, \quad e_i(p) \in \widetilde{\partial}\, \nu_i^*(p;0)$$

The mapping is upper hemicontinuous and convex, so that a fixed-point exists by Kakutani's fixed point theorem. $F_{\mathcal{E}}(p^W) = p^W$ implies $\sum_i e_i(p^W) = \mathbf{0}$, with $e_i(p^W) \in \widetilde{\partial}\, \nu_i^*(p;0)$.

To write WE as a saddle-point, use Proposition 3.1.2 to find $\bar{\alpha}_i$ such that $[\bar{\alpha}_i \nu_i]^*(p^W) = \bar{\alpha}_i \nu_i^*(p^W;0)$, i.e., e_i is the optimal decision by QL utility maximizer $\bar{\alpha}_i \nu_i$. With $\mathbf{a} = \langle \bar{\alpha}_i \rangle$, let

$$\nu_{\mathcal{E}(\mathbf{a})} = \sup \{ \sum_i \bar{\alpha}_i \nu_i(e_i) : \sum_i e_i = e \}$$

define a QL model with utilities $\bar{\alpha}_i \nu_i$. Its saddle function is

$$S_{\mathcal{E}(\mathbf{a})}(p, e) = \nu_{\mathcal{E}(\mathbf{a})} - p \cdot e - \delta_{P^\ell}(p)$$

From the construction, the saddle function has saddle point $(p^W, \mathbf{0})$.

8.2 Proof for Proposition 4.2.3

Define $\mathfrak{p}_i(p) := p\dot{\mathfrak{E}}_i$ to have the following identity.

$$p \cdot \dot{\mathfrak{E}}_i(z_i) = p\dot{\mathfrak{E}}_i \cdot z_i = \mathfrak{p}_i(p) \cdot z_i \tag{8.2.1}$$

A conjugate duality for $(\mathfrak{p}_i(p), z_i)$, similar to the one for $(p, \mathfrak{E}_i(z_i))$, can be formulated:

$$\begin{aligned}
\text{[conjugate function]: } & \delta^*_{\Delta(A_i)}(\mathfrak{p}_i(p)) = \sup\{\mathfrak{p}_i(p) \cdot z_i - \delta_{\Delta(A_i)}(z_i)\}, \\
\text{[bi-conjugacy]: } & \delta^{**}_{\Delta(A_i)}(p) = \delta_{\Delta(A_i)}(p), \\
\text{[conjugate inequality]: } & \delta^*_{\Delta(A_i)}(\mathfrak{p}_i(p)) + \delta_{\Delta(A_i)}(p) \geq \mathfrak{p}_i(p) z_i, \\
\text{[conjugate equality]: } & \delta^*_{\Delta(A_i)}(\mathfrak{p}_i(p)) + \delta_{\Delta(A_i)}(p) = \mathfrak{p}_i(p) z_i \\
& \quad \text{if and only if } \ z_i \in \partial \delta^*_{\Delta(A_i)}(\mathfrak{p}_i(p)) \text{ and } p \in \partial \delta_{\Delta(A_i)}(z_i), \\
\text{[conjugate duality]: } & \mathfrak{e}_i(z_i) \in \partial \delta^*_{\mathfrak{E}_i}(\mathfrak{T}(q)) \Longleftrightarrow z_i \in \partial \delta^*_{\Delta(A_i)}(\mathfrak{p}_i(\mathfrak{T}(q))
\end{aligned}$$

Restricting attention on $\mathfrak{T}[Q] \subset P$, we define mapping $F_N : Q \rightrightarrows Q$,

$$F_N(q) = \langle \frac{q_i + \mathfrak{p}_i(\mathfrak{T}(q))^+}{1 + \sum_{a_i} \mathfrak{p}_i(\mathfrak{T}(q))^+(a_i)} \rangle.$$

where $\mathfrak{p}_i(\mathfrak{T}(q))^+(a_i) = \max\{\mathfrak{p}_i(\mathfrak{T}(q))(a_i), 0\}$.

As in Proposition 4.1.3, Kakutani's fixed point theorem show that there is a fixed point, and $F_N(q^N) = q^N$ if and only if $\mathfrak{p}_i(\Phi(q))^+ = \mathbf{0}$.

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