Global Income Poverty Measurement with Preference Heterogeneity: Theory and Application*

Benoit Decerf,† Mery Ferrando,‡ and Natalie Naïri Quinn§

January 11, 2023

Abstract

There is growing support for monitoring global poverty using a measure that accounts for both own and relative income. We develop a theory of global poverty measurement with heterogeneous preferences over these factors and derive its implications for both the global poverty line and the poverty index. Our theory provides a welfarist foundation for the societal global poverty line. We show that standard poverty indices are not necessarily reduced when an individual escapes welfare poverty. Surprisingly, a simple modification of the (societal) headcount ratio captures the main features of our theory. For the period 1999-2015, our proposed index assesses global poverty reduction to have been 50% higher than current estimates based on the (societal) headcount ratio, while 25% lower than estimates based on the absolute headcount ratio.

JEL: I32, C43, N30.

Keywords: Global Income Poverty, Preference Heterogeneity, Welfare-Consistency, Relative Poverty, Absolute Poverty.

*Acknowledgments: We thank Peter Lanjouw, Daniel G. Malher, Berk Ozler, and Roy van der Weide, who provided useful comments on earlier versions of this paper. We are grateful to Martin Ravallion for a discussion in Namur that was at the origin of this research project. We thank all the participants at the Welfare Economics and Economic Policy webinar, the Opportunity, Mobility and Well-being Conference in Munich, the Ninth ECINEQ Meeting at LSE, SSCW 2022 in Mexico City, internal seminars at the World Bank and Tilburg University, and, in particular, Francisco Ferreira and Tim Goedeme. This work was supported by: FNRS Excellence of Science (EOS) Research project #O020918F. The findings, interpretations, and conclusions expressed in this paper are entirely those of the authors and should not be attributed in any manner to the World Bank, its affiliated organizations, or members of its Board of Executive Directors or the countries they represent. The World Bank does not guarantee the accuracy of the data included in this paper and accepts no responsibility for any consequence of their use. All remaining mistakes are, of course, our own.

†World Bank, Development Research Group. Email: bdecerf@worldbank.org.

‡Tilburg University, Department of Economics. Email: m.ferrando@tilburguniversity.edu.

§University of Oxford, Department of Economics. Email: natalie.quinn@economics.ox.ac.uk.

1 Introduction

Poverty reduction is the first Sustainable Development Goal, adopted by the United Nations in 2015. Both the design of effective poverty-reduction policies and the monitoring of progress require that poverty be measured meaningfully. There is growing support for the idea that global income poverty should be assessed with a measure that accounts for *both* own income and relative income.[1] Two related justifications for this have been proposed. First, Atkinson and Bourguignon (2001) argue that taking a global perspective requires accounting for both subsistence and social inclusion, the two functionings underpinning poverty measurement practices in developing countries and developed countries, respectively. While the real cost of subsistence is typically assumed fixed, that of social inclusion increases with standards of living and therefore depends on relative income (Smith, 1776; Townsend, 1985). Second, Ravallion and Chen (2011) argue that relative income is, like own income, an important determinant of the concept of economic welfare that is relevant for global poverty measurement.[2] These two justifications are in fact related if individuals care about both own income and relative income because they care about subsistence and social inclusion (Ravallion, 2020).

When both own income and relative income matter, the trade-off that a poverty measure makes between them becomes a key question. Altering this trade-off in practice may reverse cross-country comparisons or poverty trends, so it should not be made arbitrarily. Rather, welfare-consistency requires that this trade-off be related to individual preferences over own income and relative income.[3] Following the standard welfarist definition, an individual is welfare poor if she is worse off than at some particular reference situation (Ravallion, 1998). This definition depends on her preference. The most basic welfare-consistency property is, arguably, that a poverty measure should go down when an individual escapes welfare poverty. This *escaping-poverty* property constitutes the principal conceptual defense for using the headcount ratio when preferences are homogeneous. Clearly, a poverty measure satisfies the

[1] As recommended by Atkinson (2016), the World Bank now also reports global poverty estimates based on a measure that takes both own income and relative income into account.

[2] There is now ample evidence that relative income is an important determinant of subjective well-being (Clark and Oswald, 1996; Luttmer, 2005; Perez-Truglia, 2020).

[3] Following Ravallion (2020), we assume that an individual's preference over own income and relative income serves as a 'reduced form' for her deeper preference over her levels of nutrition and social participation (see online Appendix S1 for details). Hence, malicious aspects of the other-regarding preference are assumed to be laundered away.

escaping-poverty property only if its trade-off is related to preferences.

Current research efforts on global income poverty measurement, recently reviewed in Ravallion (2020), have two important limitations that are directly related to this trade-off.[4] The first is that this literature assumes the existence of a common welfare function, which makes the same trade-off between own income and relative income for all individuals. There are two possible interpretations for this assumption. On one hand, it could mean that all individuals make the same trade-off. However, this is in conflict with existing evidence. Previous studies have shown that even poor individuals hold heterogeneous preferences over necessities (Atkin, 2013, 2016), and a growing body of field and experimental evidence documents that social preferences are heterogeneous (Eckel and Grossman, 1998; Andreoni and Vesterlund, 2001; Blanco et al., 2011). In light of this evidence, imposition of a common trade-off appears to be paternalistic. On the other hand, one could interpret the common welfare function to represent some aggregation of heterogeneous preferences. To the best of our knowledge, our paper is the first attempt to formalise such aggregation.

The second limitation is that the literature focuses on the design of welfare-consistent global poverty *lines*.[5]

However, a poverty measure is always the combination of a poverty line and a poverty index (Sen, 1976). So far, the literature has not investigated the implications of welfare-consistency for the poverty index. These implications could matter because the choice of poverty index may impact the evaluation of the global poverty trend at least as much as the design of the global poverty line (Decerf and Ferrando, 2022).

In this paper, we develop a theory of global income poverty measurement that makes progress on these two limitations. In a framework where individuals hold heterogeneous preferences over own income and relative income, we investigate the implications of welfare-consistency not only for the poverty line but also for the poverty index.

The main assumptions of our theoretical framework are standard. Any individual is endowed with a preference over own income and relative income.

[4]There are also other important limitations not directly related to this trade-off, for example, the comparability of own income across households and across countries. Producing a comparable own income variable is challenging when dealing with heterogeneous relative prices between goods and heterogeneous preferences over these goods (Van Veelen and van der Weide, 2008; Dimri and Maniquet, 2020). In line with Atkinson and Bourguignon (2001) and Ravallion and Chen (2011), we abstract from this issue and assume that a comparable own income variable is available.

[5]Following Atkinson and Bourguignon (2001), a *global* poverty line should be consistent with a common framework applied to all countries of the world.

Relative income is defined with respect to median income, but all our results still hold when mean income is used instead. An individual is *welfare poor* when she is worse off than at some reference situation. In our framework, this reference situation is characterized by a subsistence income and a reference median income. Its specification is a normative choice exogenous to our theory; the reference median income captures the largest value of median income at which the subsistence income is sufficient for social inclusion.

We study the constraints that two requirements, one about welfare-consistency and one about fairness, impose on the definition of the poverty measure. The welfare-consistency requirement encapsulates the escaping-poverty property, which requires that the poverty measure is reduced when an individual escapes welfare poverty. A fairness requirement is called for in the presence of heterogeneous preferences to rule out counter-intuitive poverty comparisons. In particular, when comparing two individuals living in the same society, the measure should not attribute a *greater* poverty score to the individual with *greater* income on the basis that she is more sensitive to relative income. Our main results fully characterize the joint implications that these two requirements have on the global poverty line and on the poverty index (Theorems 1 and 2 in Section 5). These main results heavily rely on preference heterogeneity.

A key implication of considering heterogeneous preferences is that fair poverty measures cannot perfectly track welfare poverty. In the case of homogeneous preferences, welfare poverty could be perfectly tracked through the concept of income poverty. An individual is *income poor* when her income is smaller than the value taken by the global poverty line in her society. Meanwhile, each individual has a threshold income at which she escapes welfare poverty in her society. Being welfare poor is equivalent to being income poor when the threshold income coincides with the global poverty line. This threshold income, however, depends on preferences and thus differs across individuals with heterogeneous preferences. The difficulty is that fairness requires that all individuals living in the same society face the same global poverty line. We show that the escaping poverty property implies that all individuals who are welfare poor must be income poor, but, in the context of heterogeneous preferences, some individuals who are income poor are in fact not welfare poor.

We further show that, when preferences are heterogeneous, the escaping-poverty property is satisfied only when the global poverty line is *societal*, that is, when it is absolute in low-income countries and relative in more developed countries. The global line must always correspond to the smallest income for which no individual could be welfare poor. In countries whose median income is below the reference median income, that smallest income is the subsistence income, while in wealthier countries it varies with median income. This result provides conceptual guidance for the design of the global line and a welfarist foundation for the societal lines proposed in the literature.[6]

Under heterogeneous preferences, the escaping-poverty property also has important implications for poverty indices. In particular, the escaping-poverty property is violated by the societal headcount ratio, that is, the fraction of individuals whose income is below the global line. The violation comes from the fact that the societal headcount ratio attributes the same poverty score (equal to one) to all individuals who are income poor. The issue is that some individuals are income poor but are *not* welfare poor. Consider for instance an individual who lives in a middle-income country and whose income is smaller than the subsistence income. This individual must be welfare poor, because her country's median income is larger than the reference median income. Assume that this individual becomes only-relatively income poor, that is, her income is lifted above the subsistence income, but remains smaller than the global line. The individual remains income poor but, if she is only mildly sensitive to relative income, she has escaped welfare poverty, because she now prefers her bundle to the reference situation. However, the societal headcount ratio does not record any progress because this index attributes the same poverty score to the absolutely income poor, whose incomes are smaller than the subsistence income, and to the only-relatively income poor. We show that a slight modification of the societal headcount ratio, which we call the *hierarchical headcount ratio*, would record this progress. This modified index sums up the fraction of individuals who are absolutely income poor with the fraction of individuals who are only-relatively income poor, multiplied by a weight smaller than one. Under some assumption on the distribution of preferences, the hierarchical headcount ratio corresponds to the expected fraction of individuals who are welfare poor. Importantly, the trade-offs that this index makes between own income and relative income correspond to those characterized in our main theoretical results (Theorem 1 and 2).

We compare the empirical distribution and evolution of global income

[6] In practice, the exact design of the global line is informed by regressing national poverty lines on standards of living (Atkinson and Bourguignon, 2001; Ravallion and Chen, 2011; Jolliffe and Prydz, 2021). Virtually all global lines proposed by these authors are societal.

poverty using three different poverty measures: the absolute headcount ratio, the societal headcount ratio, and the hierarchical headcount ratio. The latter two are constructed using the same global line, namely the societal poverty line of the World Bank (2018). These three measures provide markedly different evaluations of global income poverty. For instance, over the period 1999-2015, the hierarchical headcount ratio assesses global poverty reduction to have been 50% higher than estimates based on the societal headcount ratio. The main reason behind this difference is the steep decrease in the absolute headcount ratio during this period, which has a stronger effect on the hierarchical headcount ratio than the societal headcount ratio. Indeed, unlike the societal headcount ratio, the hierarchical headcount decreases when an absolutely income poor individual becomes only-relatively income poor. Over the same period, the hierarchical headcount ratio assesses global poverty reduction to have been 25% lower than estimates based on the absolute headcount ratio, which does not account for the only-relatively income poor. Moreover, Sub-Saharan Africa accounts for 24% of global poverty in 2015 according to the societal headcount ratio, but for 36% according to the hierarchical headcount ratio. The main reason for this difference is that Sub-Saharan Africa accounts for almost 60% of global absolute income poverty.

Our theoretical results make several contributions to the literature on income poverty measurement. First, our results suggest that the assumption of preference homogeneity in the context of poverty measurement is an unnecessary modelling restriction. Indeed, we show that we can derive meaningful and ready-to-use poverty measures by aggregating heterogeneous preferences.

Second, our results provide a strong conceptual basis for discarding the societal headcount ratio and replacing it with a *hierarchical* poverty index (Decerf, 2021). We show that only hierarchical indices produce a fair and welfare-consistent aggregation of heterogeneous preferences.[7]

Third, our results provide the first welfarist foundation for global poverty lines of the societal type. Ravallion and Chen (2011) investigate the implications of a weak relativity axiom (WRA), which requires that a poverty measure is reduced when all incomes grow by the same proportion. The WRA is weaker than our welfare-consistency requirement.[8] Our results extend and improve on the results

[7]The axiomatic result of Decerf (2021) does not account for individual preferences, which are absent from his framework. Decerf (2017) merely argues that hierarchical indices yield poverty comparisons that are more in line with intuition than the comparisons associated with standard indices.

[8]Provided that the reference median income is greater than the subsistence income, a

of Ravallion and Chen (2011) in at least three crucial ways: (i) we do not assume, but rather show, that the relative segment of the global line must be linear; (ii) our conceptual results explain not only the relative segment of the global line, but also its absolute segment; and (iii) we show that, in general, the global line cannot be interpreted as providing the same utility in all countries.

Our results also contribute to the social choice literature. One of its branches aims at producing normative indices that aggregate heterogeneous preferences while satisfying the Pareto principle and fairness requirements (Fleurbaey and Maniquet, 2011). We contribute to this literature because, to the best of our knowledge, we are the first to study the implications for income poverty measurement of aggregating (other-regarding) preferences over own income and relative income.[9]

Finally, we note that from a practitioner perspective, our proposal – the implementation of the hierarchical headcount ratio to measure global poverty – is able to overcome several challenges that arise when measuring poverty in the presence of heterogeneous preferences. One such challenge is the selection of the reference bundle, which cannot be set without some subjectivity. As we show in our application, the reference bundle can simply be anchored in established poverty lines. A second challenge is the difficulty of credibly eliciting preferences. In our application we show that the extent of admitted preference heterogeneity can, again, be anchored in established poverty lines. In a sense, our proposal accounts for preference heterogeneity as much as possible under the constraint that preferences are not elicited. A third challenge relates to the communicability of the poverty measure. By relying on a simple modification of the societal headcount ratio, our proposal remains fairly simple to explain and apply.

The remainder of the paper is organized as follows. In Section 2, we provide intuitive explanations for our main theoretical results in a simplified setting and discuss how the societal headcount ratio can be adapted. In Section 3, we introduce our framework and the fairness and welfare-consistency properties considered. In Section 4, we show that these two properties clash under heterogeneous preferences and weaken the latter. In Section 5, we characterize their joint implications on

reasonable constraint.

[9] Decancq et al. (2019) also consider poverty measurement, but their framework features self-centered preferences over multidimensional goods. Treibich (2019) also considers a setting with other-regarding preferences over own income and relative income, but he studies social welfare measurement. Another difference with these authors is that we weaken Pareto while they weaken fairness.

nested sets of preferences. In Section 6, we present our application to global poverty measurement. In Section 7, we provide concluding comments.

2 Main Theoretical Findings

Any poverty measure should be reduced when a welfare poor individual escapes welfare poverty, at least when others are not negatively affected. We call this minimal welfare-consistency requirement the *escaping-poverty property*. It is arguably the most fundamental property in the poverty measurement literature. All poverty measures satisfy this property in traditional settings, that is, when only own income matters, or when both own income and relative income matter and preferences are homogeneous. This includes the headcount ratio, which captures the fraction of individuals whose income is below some poverty line. The headcount ratio has a simple interpretation, but it is well-known for violating many other properties. For instance, it does not react when the income of a poor individual increases but remains below the line (Sen, 1976). Yet, such limitations have often not been deemed sufficient for discarding this index. One important conceptual reason for using the headcount ratio is that it at least satisfies the escaping-poverty property. As we explain in this section, that is no longer the case when preferences are heterogeneous.

2.1 The Basic Framework

Let $\mathbf{y} := (y_1, \ldots, y_{n(\mathbf{y})})$ denote an income distribution, and let $\overline{y}$ denote the median income in distribution $\mathbf{y}$.[10] In line with Ravallion (2020), an individual has preferences over bundles comprising both her own income y and her relative income $y/\overline{y}$. Her preference relation can be represented by a utility function $u(y, y/\overline{y})$ that is strictly increasing in its first argument and weakly increasing in the second. For instance, an individual could have her preferences represented by a utility function in the following family:

$$u^\sigma(y, y/\overline{y}) := -\left(\frac{1}{y} + \frac{\sigma}{y/\overline{y}}\right), \tag{1}$$

[10] We use *income* as shorthand for a comparable individual monetary welfare indicator. In principle, such an indicator would be adjusted to account for household composition and individuals' needs, for example, due to disability. In practice, data limitations mean that needs-adjustments are rarely implemented.

where the parameter $\sigma \geq 0$ tunes the sensitivity to relative income. Self-centered preferences correspond to the case $\sigma = 0$, that is, to u^0.[11] This utility function is strictly increasing in relative income when $\sigma > 0$. Without loss of generality, we often write the two arguments of utility to be own income and median income, that is, we write $u(y, \overline{y})$ instead of $u(y, y/\overline{y})$, and refer to $(y, \overline{y})$ as a *bundle*.

An individual is welfare poor if she is worse off than at a reference bundle (Ravallion, 1998). The selection of this reference bundle is exogenous to our theory, and we assume it is selected by some social planner. We denote the reference bundle by $(z_a, \overline{y}^z)$, where $z_a > 0$ and $\overline{y}^z > 0$. Parameter z_a is the subsistence income, which we take in our empirical application to be the World Bank's extreme poverty line (Ferreira et al., 2016). Parameter $\overline{y}^z$ is the reference median income, which we interpret as the largest value of median income at which the subsistence income is sufficient for social inclusion. We emphasize that our definition of the welfare poor is *not* based on an exogenously given global poverty line.

Figure 1 illustrates the definition of welfare poverty. An individual is welfare poor when her bundle lies below her indifference curve passing through the reference bundle.[12] Indifference curves have non-negative slopes because relative income is positively valued. Individual 1 is not welfare poor because she is self-centered ($u_1 = u^0$) and her income is larger than the subsistence income (Figure 1.a). Individual 2 prefers the reference bundle over her bundle and is thus welfare poor (Figure 1.b).

The literature traditionally assumes that preferences are homogeneous. When the common preference is self-centered (u^0), only own income matters. The self-centered case provides the foundation for absolute poverty lines. To see this, assume that the absolute line is set at the subsistence income. With a common self-centered preference, all individuals below the subsistence income are welfare poor and all those above it are not welfare poor. In this sense, the absolute line perfectly identifies the welfare poor. In this case, the absolute headcount ratio, that is, the fraction of individuals below the subsistence income, satisfies the escaping-poverty property, because this measure is equal to the fraction of individuals who are welfare poor. When the common preference is not self-centered (e.g., $u^{\bar{\sigma}}$ for some $\bar{\sigma} > 0$), it is still possible that the poverty line perfectly identifies the welfare poor. For any given value of median income $\overline{y}$, the poverty line corresponds to

[11] In our terminology, *self-centered* has no negative connotation. It means that the individual only cares about own income.

[12] When the preference is u^σ, its indifference curves are straight lines because u^σ is ordinally equivalent to $-(u^\sigma)^{-1} = \frac{1}{\frac{1}{y} + \frac{\sigma}{y/\overline{y}}} = \frac{y}{1+\sigma\overline{y}}$.

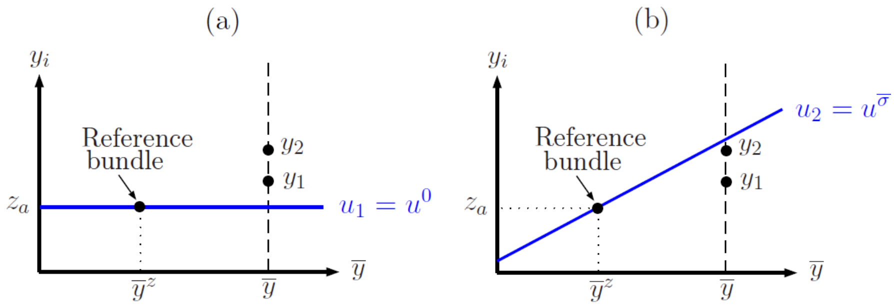


Figure 1: Definition of welfare poverty. Individual 1 is not welfare poor (a). Individual 2 is welfare poor (b).

Notes: The blue lines are indifference curves passing through the reference bundle. Individual 1's is self-centered because $u_1 = u^0$. *Individual 2 is not self-centered because* $u_2 = u^{\bar{\sigma}}$ *with* $\bar{\sigma} > 0$.

the income level $z(\overline{y})$ that provides the same utility as the reference bundle, i.e., $u^{\bar{\sigma}}(z(\overline{y}), \overline{y}) = u^{\bar{\sigma}}(z_a, \overline{y}^z)$.[13] This case provides the foundation for relative lines. The (relative) headcount ratio below the poverty line $z(\overline{y})$ satisfies the escaping-poverty property, because this measure is equal to the fraction of individuals who are welfare poor.

We depart from the literature by considering heterogeneous preferences. A society is thus characterized by a distribution-profile pair $(\mathbf{y}, \mathbf{u})$, where $\mathbf{u} := (u_1, \ldots, u_{n(\mathbf{y})})$ is the profile of utility functions. In this setting, welfare-consistent measures may provide highly counter-intuitive comparisons. Take, for instance, the measure "fraction of individuals who are welfare poor" and consider again Figure 1. Individuals 1 and 2 live in the same society and the former has a smaller income than the latter. However, individual 1 is not welfare poor because her preference is $u_1 = u^0$, whereas individual 2 is welfare poor because her preference is $u_2 = u^{\bar{\sigma}}$. In that case, the measure "fraction of individuals who are welfare poor" attributes a smaller poverty score to individual 1 (whose score is equal to zero) than to individual 2 (whose score is equal to one), because the former is less sensitive to relative income than the latter. However, this seems unfair because individual 1 has a worse objective situation than individual 2. This example illustrates that when preferences are heterogeneous, welfare-consistency and fairness may clash. In this paper, we side with fairness in order to rule out poverty measures that make counter-intuitive comparisons.

[13] We can interpret $z(\overline{y}) - z_a > 0$ as the extra income needed by a person who lives in a society with $\overline{y} > \overline{y}^z$ and earns the subsistence income z_a to also be socially included.

For this reason, we consider additive poverty indices that satisfy a classical fairness requirement. This fairness requirement has strong implications, because indices that satisfy it must treat equally any two individuals who live in the same society and earn the same income.

Therefore, the definition of these *fair* poverty indices cannot depend on the specific profile $\mathbf{u}$ of society $(\mathbf{y}, \mathbf{u})$, and hence, these indices can be implemented without eliciting individuals' preferences. But this does not mean that preferences play no role. The set U of admissible utility functions will play a central role in our analysis. Formally, a fair poverty index is

$$P_U(\mathbf{y}, \mathbf{u}) \coloneqq \frac{1}{n(\mathbf{y})} \sum_{i=1}^{n(\mathbf{y})} p(y_i, \overline{y}), \tag{2}$$

where $p(y_i, \overline{y})$ is individual i's poverty score, that is, the non-negative amount she contributes to the poverty measure. The poverty score $p(y_i, \overline{y})$ may depend on the set U, but equal treatment is guaranteed because $p(y_i, \overline{y})$ does *not* depend on i's preference u_i.

Importantly, this equal treatment extends to the poverty line, which we will also describe as *fair*. The poverty line is by definition the minimal income level for which an individual's poverty score is zero. The value of the poverty line $z(\overline{y})$ may depend on median income $\overline{y}$, but not on i's preference u_i. An individual i whose income y_i is smaller than the poverty line $z(\overline{y})$ is *income poor* and her poverty score $p(y_i, \overline{y})$ is strictly positive. An individual's income poverty status is 'objective' in the sense that it only depends on the poverty line.

An individual's welfare poverty status is (partly) 'subjective' in the sense that it depends on her preference as well as the reference bundle. When preferences are heterogeneous, an individual's income poverty status may differ from her welfare poverty status.

Before illustrating this, we show how the poverty line $z(\overline{y})$ is endogenously determined from the escaping-poverty property.

2.2 Implications for the poverty line

We show that, in the context of heterogeneous preferences, the escaping-poverty property entails that the (fair) global poverty line must be *societal*, that is, absolute in low-income societies and then relative in higher-income societies.[14] Such a

[14] The literature defines a poverty line as *absolute* when its value $z(\overline{y})$ is independent of $\overline{y}$ and as *relative* when its value $z(\overline{y})$ depends on $\overline{y}$.

societal global poverty line is illustrated by the solid red line in Figure 2. Here is the intuition. Any welfare poor individual must be attributed a strictly positive poverty score in order for the measure to be reduced when she escapes welfare poverty. Indeed, if a welfare poor individual's poverty score is zero, her poverty score cannot further decrease when she escapes welfare poverty. Hence, any bundle $(y, \overline{y})$ at which some individual could be welfare poor must be attributed a strictly positive poverty score, which by definition of the global line implies $y < z(\overline{y})$. Thus, the escaping-poverty property implies that for any value of median income $\overline{y}$, the global line $z(\overline{y})$ must be greater than or equal to the smallest income for which no individual can be welfare poor, that is, $u(z(\overline{y}), \overline{y}) \geq u(z_a, \overline{y}^z)$ for all $u \in U$. If the global line $z(\overline{y})$ is equal to the smallest income for which no individual can be welfare poor then $u'(z(\overline{y}), \overline{y}) = u'(z_a, \overline{y}^z)$ for some $u' \in U$ and we call $z(\overline{y})$ *maximal*. The definition of a maximal line depends on the set of admissible preferences U.

For the three sets of heterogeneous preferences that we consider in Section 5, the maximal line is a societal line. We illustrate this by considering an even simpler set of preferences, which only contains the two utility functions u^0 and $u^{\bar{\sigma}}$. As explained above, for given $\overline{y}$, the global line $z(\overline{y})$ corresponds to the smallest income for which no individual can be welfare poor. Graphically, this means that the global line is the upper contour of the two indifference curves through the reference bundle respectively associated with u^0 and $u^{\bar{\sigma}}$, which are drawn in Figure 1. For low-income societies, whose median income is *smaller* than the reference median income $\overline{y}^z$, the smallest income at which no one is welfare poor is determined by the preference *least* sensitive to relative income, that is, the self-centered preference u^0. Hence, this smallest income corresponds to the subsistence income z_a. Indeed, an individual with preference u^0 and own income just below z_a who lives in a low-income society is welfare poor. For higher-income societies, whose median income is *larger* than the reference median income $\overline{y}^z$, this smallest income is determined by the preference *most* sensitive to relative income, that is, $u^{\bar{\sigma}}$. Indeed, an individual with preference $u^{\bar{\sigma}}$ and income just below $z(\overline{y})$ who lives in a higher-income society is welfare poor.

We emphasize that this reasoning hinges on preference heterogeneity. Indeed, the global line is absolute in low-income countries because some individuals may not be sensitive to relative income. In turn, the global line is relative in higher-income countries because some individuals are sensitive to relative income.

The above reasoning also implies that some income poor individuals are not welfare poor. When median income is larger than the reference median income

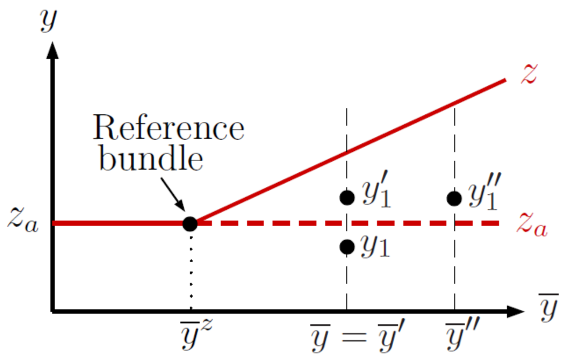


Figure 2: Societal poverty line and the distinction between income poverty and welfare poverty

Notes: The red line labelled z is the global poverty line $z(\overline{y})$ and the dashed red line is the subsistence income z_a. Individual 1 is self-centered ($u_1 = u^0$). In distribution $\mathbf{y}$, individual 1 is absolutely income poor and welfare poor. In distributions $\mathbf{y}'$ and $\mathbf{y}''$, individual 1 is only-relatively income poor, but not welfare poor.

$\overline{y}^z$, the global line is higher than the subsistence income. We describe as *absolutely income poor* the individuals whose incomes are below the subsistence income and *only-relatively income poor* the individuals whose incomes are above the subsistence income but below the global line $z(\overline{y})$. By definition, these are two mutually exclusive forms of income poverty. As illustrated in Figure 2, some individuals who are only-relatively income poor are not welfare poor. For instance, individual 1 is only-relatively income poor in distribution $\mathbf{y}'$ because $z_a < y_1' < z(\overline{y}')$, but she is not welfare poor because her preference is self-centered ($u_1 = u^0$) and $u^0(y_1', \overline{y}') > u^0(z_a, \overline{y}^z)$.

2.3 Implications for the Poverty Index

We now turn to the implications that the escaping-poverty property has on the poverty index. These implications are non-trivial in the context of heterogeneous preferences because some individuals are income poor but not welfare poor. The main implication of this property is that the poverty index is *hierarchical*: it must attribute a larger poverty score to absolutely income poor individuals than to only-relatively income poor individuals.

We begin by examining the implications of the escaping-poverty property for a value of median income $\overline{y}$ that is fixed and larger than the reference median income $\overline{y}^z$. For such $\overline{y}$, the poverty line is higher than the subsistence income and some income poor individuals are only-relatively income poor. The key observation is that some income poor individuals escape welfare poverty when their incomes increase, even if they remain only-relatively income poor. In particular, this may happen when their income surpasses the subsistence income. For instance, in Figure 2, the self-centered individual 1 is absolutely income poor, and thus welfare poor, in distribution $\mathbf{y}$, but only-relatively income poor, and thus not welfare poor, in distribution $\mathbf{y}'$, which has the same median income as $\mathbf{y}$. Hence, the escaping-poverty property is satisfied only if any only-relatively income poor individual is attributed a strictly smaller poverty score than any absolutely income poor individual living in the same society.

The above reasoning shows that the societal headcount ratio, which captures the fraction of individuals who are income poor, violates the escaping-poverty property when preferences are heterogeneous. The societal headcount ratio can be decomposed as:

$$H_S(\mathbf{y}) \coloneqq H_A(\mathbf{y}) + H_R(\mathbf{y}), \tag{3}$$

where $H_A(\mathbf{y})$ and $H_R(\mathbf{y})$ respectively denote the fraction of individuals who are absolutely income poor (which we call the absolute headcount ratio) and the fraction of individuals who are only-relatively income poor. (Our notation for specific poverty indices omits their dependence on the set U.) This decomposition reveals that the societal headcount ratio is not reduced when an absolutely income poor individual becomes only-relatively income poor, even when she escapes welfare poverty. Fundamentally, the issue is that the societal headcount ratio H_S measures not the fraction of individuals who are welfare poor but rather the fraction of individuals who are income poor.

When considering a fixed $\overline{y}$, the escaping-poverty property is satisfied by the *expected* fraction of individuals who are welfare poor. Assume that income distributions are observable but preferences are not observable. For some belief that an observer may hold with respect to the distribution of preferences, we denote by $\mathbb{EH}(\mathbf{y})$ the fraction of individuals that the observer expects to be welfare poor in distribution $\mathbf{y}$. For instance, if she believes there is a 50% probability that any individual has preference u^0 and a 50% probability that the individual has preference $u^{\bar{\sigma}}$, then for any distribution $\mathbf{y}$ with $\overline{y} > \overline{y}^z$, $\mathbb{EH}$ corresponds to the index $H_S^{1/2}$ defined as

$$H_S^{1/2}(\mathbf{y}) \coloneqq H_A(\mathbf{y}) + \frac{1}{2} H_R(\mathbf{y}), \tag{4}$$

where the poverty score of any absolutely income poor individual is one and that of any only-relatively income poor is one-half. Index $H_S^{1/2}$ is reduced when a self-centered individual escapes welfare poverty. The reason is that her income must surpass the subsistence income, implying her poverty score is reduced by one half.

Index $H_S^{1/2}$ is arguably too simplistic since it will also violate the escaping-poverty property as soon as some individuals have a sensitivity to relative income that is intermediate between those of u^0 and $u^{\bar{\sigma}}$. Indeed, an individual with preference $u^{\bar{\sigma}/2}$ can have two different welfare poverty statuses for two different bundles that both yield only-relatively income poverty status. For instance, this individual could be welfare poor when consuming the bundle of individual 1 but not welfare poor when consuming the bundle of individual 2 (see Figure 1). She would thus escape welfare poverty when she moves from the former bundle to the latter, even if the index $H_S^{1/2}$ attributes the same poverty score to both.

We therefore favor another index that better reflects more realistic distributions of preferences. We call this second index the *hierarchical headcount ratio* (HH_S). Index HH_S is again defined as the sum of the fraction of absolutely income poor and the fraction of only-relatively income poor individuals multiplied by some weight smaller than one. The particularity of HH_S is that its weight is endogenous to the income distribution. For the societal poverty line $z(\overline{y})$, the *hierarchical headcount ratio* is defined as:

$$HH_S(\mathbf{y}) \coloneqq H_A(\mathbf{y}) + \omega(\mathbf{y})H_R(\mathbf{y}), \tag{5}$$

where the endogenous weight $\omega(\mathbf{y}) \in [0, 1]$ has the following linear expression:

$$\omega(\mathbf{y}) \coloneqq \frac{z(\overline{y}) - \hat{y}^R}{z(\overline{y}) - z_a}$$

for $\overline{y} > \overline{y}^z$, where $\hat{y}^R$ is the average income among only-relatively income poor individuals;[15] $\omega(\mathbf{y}) \coloneqq 0$ for $\overline{y} \leq \overline{y}^z$. The closer the average income among those only-relatively income poor is to z_a (respectively $z(\overline{y})$), the closer their weight is to one (respectively zero). Contrasting Equations (3) and (5) reveals that poverty as measured by HH_S always lies between the absolute headcount ratio and the

[15] $\hat{y}^R \coloneqq \frac{1}{n(\mathbf{y})H_R(\mathbf{y})} \sum_{i \in N_R(\mathbf{y})} y_i$, where $N_R(\mathbf{y})$ is the set of only-relatively income poor individuals.

societal headcount ratio. More precisely, we have

$$\begin{aligned} H_A(\mathbf{y}) = HH_S(\mathbf{y}) = H_S(\mathbf{y}) \qquad & \text{if} \qquad \overline{y} \leq \overline{y}^z, \\ H_A(\mathbf{y}) \leq HH_S(\mathbf{y}) \leq H_S(\mathbf{y}) \qquad & \text{if} \qquad \overline{y} > \overline{y}^z. \end{aligned} \tag{6}$$

Under some assumption on the distribution of preferences, index HH_S corresponds to the expected fraction of individuals who are welfare poor when $\overline{y} > \overline{y}^z$. The probability that an individual who is not income poor is welfare poor is zero, because the poverty line corresponds to the smallest income for which no individual can be welfare poor. The probability that an absolutely income poor individual is welfare poor is one because her own income is smaller than z_a and her relative income is smaller than $z_a/\overline{y}^z$. The assumption under which index HH_S corresponds to $\mathbb{EH}$ is that the probability that an only-relatively income poor individual is welfare poor increases *linearly* between zero and one as her income decreases from the poverty line to the subsistence income. Under this assumption, the probability that any individual i is welfare poor corresponds to the poverty score that HH_S attributes to i, which is:

$$p^{HH_S}(y_i, \overline{y}) := \begin{cases} 1 & \text{if } \ y_i < z_a, \\ \frac{z(\overline{y}) - y_i}{z(\overline{y}) - z_a} & \text{if } \ z_a \leq y_i < z(\overline{y}). \end{cases}$$

As an illustration, we show in online Appendix S2 that, for a certain probability distribution on the class of utility functions u^σ, index HH_S is equal to $\mathbb{EH}$ for all distributions $\mathbf{y}$ such that $\overline{y} > \overline{y}^z$.[16]

We consider now the implication of the escaping-poverty property when the value of median income $\overline{y}$ varies. The key observation is that self-centered individuals escape welfare poverty when their income surpasses the subsistence income, even if median income is simultaneously increased. For instance, in Figure 2, the self-centered individual 1 is absolutely income poor in distribution $\mathbf{y}$, and thus welfare poor. In contrast, individual 1 is only-relatively income poor in distribution $\mathbf{y}''$, and thus not welfare poor. Hence, the escaping-poverty property is satisfied only if any only-relatively income poor individual is attributed a strictly smaller poverty score than any absolutely income poor individual, even if they live in societies with different values of median income.

[16] We also discuss in online Appendix S2 the relationship between our theory and the literature on the fuzzy measurement of poverty.

The view that only-relatively income poor individuals should always contribute a smaller amount to poverty than absolutely income poor individuals seems to be largely shared, as survey evidence suggests (Decerf and Ferrando, 2022). We have thus shown that the escaping-poverty property provides a welfarist foundation for this view.

This second implication of the escaping-poverty property is violated by the societal headcount ratio but satisfied by the hierarchical headcount ratio. Index H_S violates it because H_S attributes a poverty score equal to one to all income poor individuals. Index HH_S satisfies it because HH_S attributes a poverty score equal to one to absolutely income poor individuals and a smaller poverty score to only-relatively income poor individuals.

We conclude this section with three remarks. First, index HH_S need not always satisfy the escaping-poverty property. Indeed, HH_S violates that property if some absolutely income poor individuals are not welfare poor.[17] Thus, even if index HH_S comes closer than index H_S to satisfying the escaping-poverty property, it does not fully do so.[18] Unfortunately, this is the price to pay for considering additive indices that do not treat individuals differently based on their sensitivity to relative income. No index defined by Equation (2) fully satisfies the escaping-poverty property.[19] Importantly, the interpersonal comparisons across different societies made by index HH_S are in line with our theory. Our main theoretical results (see Theorems 1 and 2 in Section 5) show that the trade-offs that index HH_S makes between own income and relative income correspond to those implied by a fair and welfare-consistent aggregation of heterogeneous preferences.

Second, our results do not require that some individuals be completely self-centered. It is, in fact, sufficient that some individuals are not affected by relative income when their income is below the subsistence income z_a. Alternatively, it is also sufficient that the sensitivity to relative income is arbitrarily small.

Third, some characteristics of the hierarchical headcount ratio, in particular its bounds as stated in Equation (6), have an interesting connection with the

[17]This could, for instance, be the case for an individual with preference $u^{\bar{\sigma}}$ when the value of median income is smaller than the reference median income $\overline{y}^z$. Such individual could escape welfare poverty without changing her absolute income poverty status, while index HH_S attributes a poverty score equal to one to all absolutely income poor individuals.

[18]As we show in online Appendix S3, HH_S satisfies a weak version of the escaping-poverty property that H_S violates.

[19]Even the expected fraction of poor $\mathbb{EH}(\mathbf{y})$ violates the second implication of the escaping-poverty property when some absolutely income poor individuals are not welfare poor under some preference u^{σ}. Indeed, one can then find two bundles, $(y, \overline{y})$ and $(y', \overline{y}')$, with $y < z_a < y'$ and $\overline{y} < \overline{y}^z < \overline{y}'$ such that a self-centered individual is welfare poor in $(y, \overline{y})$ but not in $(y', \overline{y}')$, whereas an individual with preference u^{σ} is welfare poor in $(y', \overline{y}')$ but not in $(y, \overline{y})$.

proposal in Ravallion and Chen (2019) to use lower and upper bounds on the fraction of individuals who are welfare poor. Ravallion and Chen assume the existence of a *common* welfare function that is unknown. They observe that the exact shape of the global poverty line depends on the specification of this common welfare function, whose exact sensitivity to relative income σ is unknown but lies in an interval $[0, \bar{\sigma}]$. For median income larger than the reference median income $\overline{y}^z$, the poverty line must be in an income range bounded below by z_a and bounded above by the income level associated with the maximal sensitivity to relative income ($\bar{\sigma}$). This implies two bounds on the fraction of individuals who are welfare poor, namely H_A and H_S. The main conceptual difference with our approach is that we assume preference heterogeneity, whereas they assume a common welfare function that is unknown. As a consequence, the weight ω that should be given to the only-relatively income poor is endogenous in our theory whereas it is unknown in Ravallion and Chen. Another difference is that our framework allows characterization of the trade-offs that the measure should make *below* the global line (Theorems 1 and 2), which is absent in Ravallion and Chen.

3 The Complete Framework

In Sections 3, 4 and 5, we present the complete theoretical results. These sections stand alone, so the reader can skip to the empirical application (Section 6) if not interested in the technical findings.

3.1 Income Distributions and Preference Profiles

An income distribution, $\mathbf{y} := (y_1, \ldots, y_{n(\mathbf{y})})$, is a list of non-negative incomes. Let $N(\mathbf{y}) := \{1, \ldots, n(\mathbf{y})\}$ denote the set of individuals in distribution $\mathbf{y}$, where $n(\mathbf{y}) \in \mathbb{N}$. Let $\overline{y}$ denote the median income in distribution $\mathbf{y}$. (As explained in online Appendix S5, all of our results still hold when $\overline{y}$ denotes mean income instead of median income.) When we wish to emphasize the median income in a particular distribution $\mathbf{y}$, we write $\overline{\mathbf{y}} = \overline{y}$.

We assume that every individual i has a complete, transitive, and continuous preference that can be represented by the utility function $u_i(y_i, y_i/\overline{y})$. We often drop subscript i when discussing points that are not specific to a given individual, for example, writing $u(y, y/\overline{y})$ instead of $u_i(y_i, y_i/\overline{y})$. We impose two (ordinal) monotonicity assumptions on preferences. First, utility functions are strictly increasing in own income when holding relative income constant; that is,

$\partial_1 u > 0$ whenever u is differentiable. It follows that, when own income and the median income are multiplied by a common factor $\delta > 1$, we have $u(\delta y, y/\overline{y}) \geq u(y, y/\overline{y})$, and the inequality is strict when $y > 0$. Second, utility functions are weakly increasing in relative income when holding own income constant; that is, $\partial_2 u \geq 0$ whenever u is differentiable. (Our results also hold if u is strictly increasing in its second argument.) These two monotonicity assumptions together imply that utility functions are strictly increasing in own income when holding median income constant.

We find it convenient to represent preferences with utility functions, but our theory only relies on ordinal preferences. Hence, our terminology uses utility functions and preferences interchangeably, even if they are different formal objects.

Let U^B denote the set of individual utility functions representing preferences that satisfy these basic restrictions. Some of our results are based on narrower sets of preferences. Let $U \subseteq U^B$ denote a generic subset of utility functions.

Let $\mathbf{u} := (u_1, \ldots, u_{n(\mathbf{u})})$ denote a profile of utility functions (or preferences profile). For a given set U, the domain of distribution-profile pairs $(\mathbf{y}, \mathbf{u})$ is $\mathcal{X}_U := \bigcup_{n \in \mathbb{N}} Y^n \times U^n$ where $Y^n := \left\{\mathbf{y} \in \mathbb{R}^n_+ | \overline{y} \geq z_a\right\}$. The mild restriction $\overline{y} \geq z_a$ is necessary for our results.[20] We respectively denote the set of bundles that an individual can consume and the subset of bundles with income smaller than z_a by

$$X := \{(y, \overline{y}) \in \mathbb{R}_+ \times [z_a, \infty)\}$$
$$X_A := \{(y, \overline{y}) \in [0, z_a) \times [z_a, \infty)\},$$

where the subscript A reflects that the frontier of this subset is the 'absolute' subsistence income z_a.

3.2 Definition of the Poor

As is standard in poverty measurement (Ravallion, 1998; Alkire and Foster, 2011), the identification of the poor is based on a reference bundle, specified exogenously by a social planner. The reference bundle $(z_a, \overline{y}^z) \in \mathbb{R}_{++} \times \mathbb{R}_{++}$ comprises a subsistence income and the maximal value of median income at which the social planner considers the subsistence income sufficient for social inclusion. It is subject to the restriction that $z_a/\overline{y}^z \leq 1$, which is needed for our results. We consider this a weak restriction, as if it were violated then the social planner would consider all

[20] Observe that although our theoretical results require that $\overline{y} \geq z_a$, the poverty indices characterized can readily be applied for countries with $\overline{y} < z_a$. In our sample, there are 11 countries for which $\overline{y} < z_a$ in 2015.

individuals socially excluded in the equal distribution $(z_a, \ldots, z_a)$.[21] Furthermore, as is standard in frameworks with heterogeneous preferences (Decancq et al., 2019; Dimri and Maniquet, 2020), an individual is deemed welfare poor if she is worse off than at the reference bundle, that is, if she prefers the reference bundle over the bundle she consumes. We have two remarks with regard to this definition. First, this is the standard 'welfarist' definition of poverty, according to which an individual is welfare poor if her well-being is below that associated with some reference bundle (Ravallion, 1998). Second, the reference bundle is the cornerstone of interpersonal comparisons. Indeed, any two individuals consuming the reference bundle are not welfare poor, even when they have different preferences.

The set of individuals who are welfare poor in the distribution-profile pair $(\mathbf{y}, \mathbf{u})$ is denoted by $Q(\mathbf{y}, \mathbf{u}) \coloneqq \{i \in N(\mathbf{y}) | u_i(y_i, \overline{y}) < u_i(z_a, \overline{y}^z)\}$. For an individual with utility function u, we denote by

$$X_Q(u) \coloneqq \{(y, \overline{y}) \in X | u(y, \overline{y}) < u(z_a, \overline{y}^z)\}.$$

the set of bundles whose consumption leaves her in welfare poverty. The union of these sets of bundles for a given set of preferences U is denoted by $X_Q(U) \coloneqq \bigcup_{u \in U} X_Q(u)$.

For some bundles, whether or not an individual is welfare poor does not depend on her preference. For instance, all individuals whose income is smaller than z_a when median income is larger than $\overline{y}^z$ are welfare poor. In turn, all individuals whose bundle is not in $X_Q(U^B)$ are not welfare poor, regardless of their preferences.

3.3 Poverty Indices

A poverty index is a function, $P_U : \mathcal{X}_U \to \mathbb{R}$, that represents a poverty ranking on $\mathcal{X}_U$. That is, $P_U(\mathbf{y}, \mathbf{u}) \geq P_U(\mathbf{y}', \mathbf{u}')$ indicates that there is (weakly) more poverty in $(\mathbf{y}, \mathbf{u})$ than in $(\mathbf{y}', \mathbf{u}')$. This definition of the poverty index allows cross-country poverty comparisons to be made, since the index is able to compare across different preference profiles.

We restrict our attention to a family of additive indices that sum up individual poverty scores. For our purposes, an individual's poverty score depends solely on her utility function and her bundle. We allow an individual's poverty score to

[21] Observe that the reference bundle $(z_a, \overline{y}^z) = (1.9, 1.8)$ associated with the societal line of the World Bank slightly violates this restriction. We do not view this as a major issue, though, given that our main objective in the empirical section is to illustrate the impact of replacing the societal headcount ratio with the hierarchical headcount ratio.

depend on her utility function at this point for the sake of generality. However, our results will preclude such dependence.

Definition 1 (Additive index)**.** *Given any $U \subseteq U^B$, we say that $P_U : \mathcal{X}_U \to \mathbb{R}$ is an additive index if*

$$P_U(\mathbf{y}, \mathbf{u}) := \frac{1}{n(\mathbf{y})} \sum_{i=1}^{n(\mathbf{y})} p_{u_i}(y_i, \overline{y}), \tag{7}$$

where for every $u \in U$, the poverty score function $p_u : X \to \mathbb{R}$ is well-defined on X and continuous on $X_Q(u)$.

This definition introduces a poverty score function $p : X \times U \to \mathbb{R}$ such that $p : (y, \overline{y}, u) \mapsto p_u(y, \overline{y})$, where, in line with the definition of utility functions, the poverty score of an individual depends on the incomes of other individuals only through median income $\overline{y}$. While this is a strong requirement, virtually all applications resort to such additive indices. More fundamentally, Decerf (2021) shows in a similar framework that axioms à la Foster and Shorrocks (1991) similarly imply such an additive expression.

Nevertheless, besides imposing a mild form of continuity, the definition of an additive index places no constraint on the way in which the poverty score function compares different bundles.[22] This is important, because the definition of the poverty score function governs the trade-offs that the measure makes between own income and relative income, and thus also the interpersonal comparisons across societies with different median incomes.

We constrain the poverty score function by imposing two properties that encapsulate fairness and welfare-consistency requirements. First, our welfare-consistency requirement adapts the Pareto principle to our poverty measurement setting. In a social welfare measurement setting, the Pareto principle requires that social welfare improves when the utility of all individuals increases. In our setting, *Pareto* requires that poverty cannot increase when the utility of all individuals increases.[23] Moreover, poverty is strictly reduced when the utility of some welfare poor individual is strictly improved.

[22] The index "fraction of individuals who are welfare poor" is not ruled out by such mild continuity.

[23] It would also be natural to require that poverty cannot increase when the utility of all *welfare poor* individuals increases. This stronger version of Pareto would only exacerbate the clash with the fairness property.

Axiom 1 (*Pareto*)**.** *For all* $(\mathbf{y},\mathbf{u}),(\mathbf{y}',\mathbf{u}) \in \mathcal{X}_U$ *such that* $n(\mathbf{y}) = n(\mathbf{y}')$*, if* $u_i(y'_i,\overline{y}') \geq u_i(y_i,\overline{y})$ *for all* $i \in N(\mathbf{y})$*, then* $P_U(\mathbf{y}',\mathbf{u}) \leq P_U(\mathbf{y},\mathbf{u})$*. If, in addition,* $u_\ell(y'_\ell,\overline{y}') > u_\ell(y_\ell,\overline{y})$ *for some* $\ell \in Q(\mathbf{y},\mathbf{u})$*, then* $P_U(\mathbf{y}',\mathbf{u}) < P_U(\mathbf{y},\mathbf{u})$*.*

Pareto encapsulates a version of the escaping-poverty property, which we discussed in Section 2. Indeed, when a welfare poor individual escapes welfare poverty, her utility is strictly increased. In that case, *Pareto* requires that the index is strictly reduced, at least when no other individual is made worse off.

Second, our fairness requirement adapts the Domination principle to our setting. When measuring social welfare, the Domination principle requires that social welfare is improved when the bundle of each individual is improved according to all relevant preferences.[24] When measuring poverty, we are interested in the trade-offs that welfare poor individuals make. In our setting, *Domination* requires that if the bundle of each individual is improved according to all utility functions under which they are welfare poor in the final distribution (if any), then poverty cannot increase, regardless of the exact preferences profiles.

Axiom 2 (*Domination*)**.** *For all* $(\mathbf{y},\mathbf{u}),(\mathbf{y}',\mathbf{u}') \in \mathcal{X}_U$ *such that* $n(\mathbf{y}) = n(\mathbf{y}')$*, if* $u(y'_i,\overline{y}') \geq u(y_i,\overline{y})$ *for all* $i \in N(\mathbf{y})$ *and all* $u \in U$ *such that* $(y'_i,\overline{y}') \in X_Q(u)$*, then* $P_U(\mathbf{y}',\mathbf{u}') \leq P_U(\mathbf{y},\mathbf{u})$*.*

In contrast to *Pareto*, *Domination* does not hold the preference profile fixed. *Domination* is a strong axiom that, we will show, implies that two different individuals with the same own income in the same society must be attributed the same poverty score, even if they have different sensitivities to relative income (Proposition 1). *Domination* thus prevents an individual who is particularly sensitive to relative income being attributed a *larger* poverty score than another individual who lives in the same society and has a *smaller* income.[25]

We illustrate in Figure 3 how the *Domination* axiom works. Assume that there are only two different utility functions in set U; that is, $U = \{u, u^0\}$. We show that *Domination* implies that poverty in distribution $\mathbf{y}'$ cannot be larger than in distribution $\mathbf{y}$. First, the bundles of individuals 2 and 3 are irrelevant for the comparison because these individuals cannot be worse off than at the

[24] The Domination principle was originally called the Intersection principle in Sen (1985).

[25] We emphasize that the implications of *Domination* are satisfied by all the standard poverty measures that satisfy the assumptions of Ravallion and Chen (2011). But *Domination* also implies that the poverty measure must focus on the preferences of individuals who are welfare poor. As a result, all bundles that cannot be consumed by an individual who is welfare poor can, without loss of generality, be attributed a poverty score equal to zero.

.

reference bundle, regardless of whether their utility functions are u or u^0.[26] Second, individual 1 is worse off than at the reference bundle in both $\mathbf{y}$ and $\mathbf{y}'$, regardless of whether her utility function is u or u^0. So, the poverty comparison of $\mathbf{y}'$ and $\mathbf{y}$ depends on the comparison of individual 1's bundles under each of these two utility functions. *Domination* implies that poverty is no greater in $\mathbf{y}'$ because bundle $(y'_1, \overline{y}')$ yields a higher utility than bundle $(y_1, \overline{y})$ for *both* utility functions (u or u^0). Observe that the axiom would have remained silent if the two utility functions had implied opposite comparisons of the bundles of individual 1 (which cannot be the case in Figure 3 since individual 1's own income and relative income are both greater in $\mathbf{y}'$).

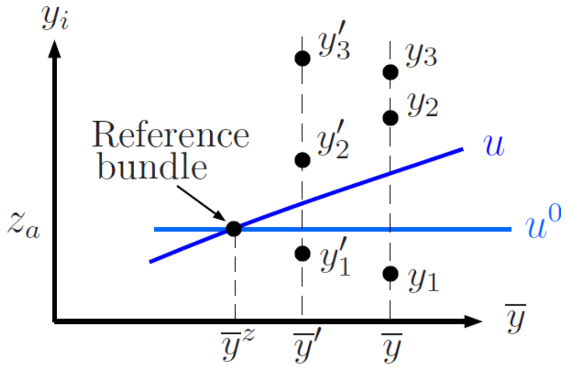


Figure 3: Comparing poverty in distributions $\mathbf{y}$ and $\mathbf{y}'$ based on *Domination*.

Notes: The blue curves are indifference curves. Under Domination, poverty cannot be larger in $\mathbf{y}'$ *than in* $\mathbf{y}$ *when* $U = \{u, u^0\}$.

4 Baseline Results and Impossibility

This section prepares the stage for our main results (presented in Section 5).

4.1 Fair Additive Poverty Indices

In Proposition 1 we show that any additive index satisfying *Domination* must be a *fair additive index.*

The main characteristic of a fair additive index is that the poverty score of an individual does not depend on her preference. As a result, fair additive indices do not make the counter-intuitive interpersonal comparisons discussed in the Introduction. Also, a fair additive index is based on a global poverty line that

[26]Formally, we have that $(y_i, \overline{y}), (y'_i, \overline{y}') \notin X_Q(U)$ for all $i \in \{2, 3\}$.

cannot admit an individual's preference among its arguments. Mathematically, the value of the global line is given by a function $z : [z_a, \infty) \to \mathbb{R}_+$ whose sole argument is the median income $\overline{y}$. We sometimes call function z the global line, even if the global line $z(\overline{y})$ is formally a different object than function z. Function z partitions the set of bundles between those below the global line,

$$X_z := \{(y, \overline{y}) \in X | y < z(\overline{y})\},$$

and those on and above the global line, $X \backslash X_z$. The poverty score of bundles below the global line is strictly positive and the poverty score of bundles above the global line is zero.[27]

Definition 2 (Fair additive index)**.** *Given any $U \subseteq U^B$, we say that $P_U : \mathcal{X}_U \to \mathbb{R}$ is a fair additive index if*

$$P_U(\mathbf{y}, \mathbf{u}) := \hat{k} + \frac{1}{n(\mathbf{y})} \sum_{i=1}^{n(\mathbf{y})} p(y_i, \overline{y}) \tag{8}$$

for some $\hat{k} \in \mathbb{R}$ where the (degenerate)[28] poverty score function $p : X \to \mathbb{R}$ is such that, for some continuous function $z : [z_a, \infty) \to \mathbb{R}_+$ such that $X_z \subseteq X_Q(U)$, we have: (i) $p(y, \overline{y}) = 0$ on $X \backslash X_z$, (ii) $p(y, \overline{y}) > 0$ on X_z, (iii) p is continuous on X_z, (iv) p is weakly decreasing in its first argument on X_z, and (v) p is weakly increasing in its second argument on X_z.

Proposition 1. *Given any $U \subseteq U^B$, an additive index P_U satisfies Domination only if P_U is a fair additive index.*

Proof. See online Appendix S6. ■

Proposition 1 may not be surprising, but it has fundamental implications. *Domination* implies that the measure is based on some *preference-independent* global line, which is the frontier of an *objectively* defined income poverty status. This is in contrast to the welfare poverty status, which has an element of subjectivity because it depends on the individual's preference.

[27] The parameter $\hat{k}$ in Eq. (8) permits the anchoring to zero without any loss of generality.

[28] We call p a degenerate poverty score function because this object is formally distinct from a poverty score function p_u (see Definition 1).

Definition 3 (Preference-independent global poverty line and income poverty status). *Given any $U \subseteq U^B$, we say that the additive index P_U is based on a preference-independent global poverty line if there is a function $z : [z_a, \infty) \to \mathbb{R}_+$ such that for all $u \in U$, we have $p_u(y, \overline{y}) > 0$ for all $(y, \overline{y}) \in X_z$ and $p_u(y, \overline{y}) = 0$ for all $(y, \overline{y}) \in X \backslash X_z$. If this is the case, we say that any individual with bundle $(y, \overline{y})$ is income poor if $(y, \overline{y}) \in X_z$, absolutely income poor if $(y, \overline{y}) \in X_z \cap X_A$, and only-relatively income poor if $(y, \overline{y}) \in X_z \backslash X_A$.*

We emphasize that we do not assume that the global line is preference-independent, but this is rather a characteristic of indices satisfying *Domination*. For instance, the additive index "fraction of individuals who are welfare poor" is not based on a preference-independent global line. Indeed, as soon as preferences are heterogeneous, this index requires a non-degenerate poverty score function; that is, the poverty score of an individual must depend on her preference.

The definition of a fair additive index does not constrain the trade-offs that the (degenerate) poverty score function makes between own income and relative income. In the remainder of our theory, we investigate how welfare-consistency requirements tie these trade-offs to those embedded in individual preferences.

4.2 Benchmark: Homogeneous Preferences

As a benchmark, we consider the case for which all individuals have the same preference. It is well-known that, when measuring social welfare, the Pareto principle is compatible with the Domination principle when preferences are homogeneous (Fleurbaey and Trannoy, 2003). This is also the case in our setting where we measure poverty. Proposition 2 shows that any additive index satisfying the two properties of fairness and welfare-consistency is such that its poverty score function is a (negative) representation of the common preference.

Proposition 2. *Given any $\{u\} \subset U^B$, the additive index $P_{\{u\}}$ satisfies Domination and Pareto if and only if $P_{\{u\}}$ is a fair additive index with a global line z such that $X_z = X_Q(\{u\})$ and a poverty score function p such that for all $(y, \overline{y}), (y', \overline{y}') \in X_z$,*

$$p(y', \overline{y}') \leq p(y, \overline{y}) \;\Leftrightarrow\; u(y', \overline{y}') \geq u(y, \overline{y}).$$

Proof. See online Appendix S7. ∎

We emphasize two implications of Proposition 2 for the properties of the global poverty line. First, the global line corresponds to the indifference curve passing through the reference bundle (see Figure 4). That is, the global line yields, for each individual (separately), the same utility in all countries.[29] Second, the global line yields a *perfect identification* of the welfare poor; that is, all individuals below the global line are welfare poor and all individuals above the global line are not welfare poor. This is because all individuals hold the common utility function.

The trade-offs that the measure makes between own income and relative income can be graphically illustrated by means of its iso-poverty-score map (see Figure 4). An iso-poverty-score map is a collection of iso-poverty-score curves, which are sets of bundles that yield a constant poverty score. The global line is one iso-poverty-score curve, or, more precisely, it is the frontier of a 'thick' iso-poverty-score curve. Below the global line, iso-poverty-score curves exactly correspond to indifference curves. The two properties (fairness and welfare-consistency) completely characterize the trade-offs made by the measure; that is, the comparison of any two bundles is determined.

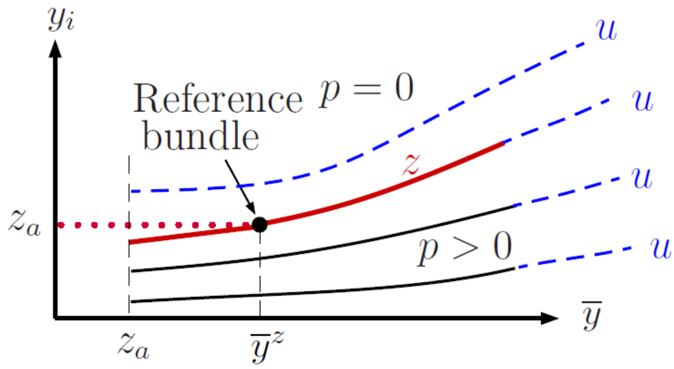


Figure 4: Iso-poverty-score curves under homogeneous preferences

Notes: Plain black curves are iso-poverty-score curves. Dashed blue curves are indifference curves. The thick red curve is the global line. Under homogeneous preferences, iso-poverty-score curves correspond to indifference curves below the global line.

4.3 General Impossibility and Weak Pareto

In many settings with heterogeneous preferences, the Pareto principle conflicts with the Domination principle (Fleurbaey and Trannoy, 2003; Brun and

[29]Our assumption of homogeneous ordinal preferences does not imply interpersonal level-comparability of utility.

Tungodden, 2004). This conflict also arises in our setting, as we show in Proposition 3. Intuitively, the reason is that the Pareto principle requires that the poverty score function (negatively) represents individual preferences, while the Domination principle requires that the poverty score function does not depend on preferences.

We say that the set U is *heterogeneous* if there exist two $u, u' \in U$ and some $(y, \overline{y}) \in X$ with $\overline{y} \geq \overline{y}^z$ such that $(y, \overline{y}) \in X_Q(u)$ and $(y, \overline{y}) \notin X_Q(u')$.[30]

Proposition 3. *Given any heterogeneous $U \subseteq U^B$, no additive index P_U satisfies Domination and Pareto.*

Proof. See Appendix A2. ∎

When confronted with similar incompatibilities in other settings, authors have taken one of either two routes: weaken the Pareto principle or weaken the Domination principle.[31] We follow the former route. We believe that poverty indices violating *Domination* would not garner much support, because they make interpersonal comparisons that many would consider counter-intuitive. This route also has a non-negligible pragmatic advantage. Poverty indices satisfying the Domination principle are easy to implement in practice because they do not require elicitation of individual preferences (Proposition 1). Therefore, our objective for the remainder of this section is to identify the 'lightest' weakening of *Pareto* that allows us to escape the impossibility (Proposition 3).

The incompatibility is less deep in our setting with other-regarding preferences, because *Pareto* has limited 'bite'. The reason is that there is never a unanimous improvement when the median income is reduced. When median income is reduced, at least one individual is made worse off, because her own income is decreased, while her relative income does not increase.

Lemma 1. *Given any $U \subseteq U^B$, for all $(\mathbf{y}, \mathbf{u}), (\mathbf{y}', \mathbf{u}) \in \mathcal{X}_U$ with $N(\mathbf{y}) = N(\mathbf{y}')$ and $\overline{y} < \overline{y}'$, there exists $j \in N(\mathbf{y})$ for whom $u_j(y'_j, \overline{y}') > u_j(y_j, \overline{y})$.*

[30] Our definition of a heterogeneous set U is not the complement of that of an homogeneous set $\{u\}$. For instance, set $\{u, u'\}$ with $u \neq u'$ does not meet our definition of an heterogeneous set when u and u' share the same indifference curve through the reference bundle. Our definition is convenient for our results.

[31] For instance, in social welfare measurement settings, Fleurbaey and Maniquet (2006, 2011) and Decancq et al. (2015) weaken the Domination principle, while Sprumont (2012) weakens the Pareto principle. Sprumont (2012) defines a *Consensus* axiom, whose precondition for recording a social improvement is that everybody finds that everyone's bundle is better in the new allocation. Consensus is a rather 'heavy' weakening because Pareto's precondition only requires that everybody finds her own bundle better in the new allocation.

Proof. See Appendix A3. ■

The incompatibility presented in Proposition 3 arises because all individuals may prefer a distribution that has more individuals below the global line. To see this, assume that some distribution $\mathbf{y}$ has no individual below the global line $z(\overline{y})$. There may exist another unanimously preferred distribution $\mathbf{y}'$ in which one individual is below the global line $z(\overline{y}')$. For instance, this may happen when the global line $z(\overline{y}')$ is higher than $z(\overline{y})$, because median income is higher in $\mathbf{y}'$ than in $\mathbf{y}$. In that case, a self-centered individual j who is not welfare poor may prefer her bundle $(y_j', \overline{y}')$ below the global line $z(\overline{y}')$ over her bundle $(y_j, \overline{y})$ above the global line $z(\overline{y})$. *Pareto* requires that the poverty index cannot be larger for the unanimously preferred distribution $\mathbf{y}'$, even if j is income poor in $\mathbf{y}'$ but not in $\mathbf{y}$. In contrast, *Domination* requires that the measure be based on a fair additive index, which attributes a poverty score equal to zero to individuals who are not income poor and a positive poverty score to individuals who are income poor. Therefore, the poverty index must be strictly larger for distribution $\mathbf{y}'$. Thus, the incompatibility arises because *Pareto* requires that the poverty index is reduced *even when* the unanimously preferred situation features more individuals below the global line.

In order to escape the impossibility, we consider a weak version of *Pareto* that remains silent when the number of income poor individuals is increased. To do this, we need to add a precondition requiring that individuals who are not welfare poor in the initial distribution do not fall below the global line in the final distribution. Of course, the global line is not a primitive of our framework and we can thus not express this precondition by referring to the global line. Fortunately, we can rely on the fact that individuals who are not income poor cannot fall below the global line when their income grows in the same proportion as the median income. Formally, *Weak Pareto* is a weakening of *Pareto* that adds a precondition for individuals who are not welfare poor in the initial distribution. Their rate of income growth is required to be non smaller than the rate of growth of the median income.[32]

Axiom 3 (*Weak Pareto*)**.** *For all* $(\mathbf{y}, \mathbf{u}), (\mathbf{y}', \mathbf{u}) \in \mathcal{X}_U$ *such that* $n(\mathbf{y}) = n(\mathbf{y}')$*, if* $u_i(y_i', \overline{y}') \geq u_i(y_i, \overline{y})$ *for all* $i \in N(\mathbf{y})$ *and* $y_j' \geq \frac{\overline{y}'}{\overline{y}} y_j$ *for all* $j \notin Q(\mathbf{y}, \mathbf{u})$*, then*

[32] If this weakening is deemed too favourable to individuals who are not welfare poor, one could restrict the additional precondition to the subset of individuals among them whose income is no greater than the median income in the final distribution, that is, for all $j \notin Q(\mathbf{y}, \mathbf{u})$ with $y_j \leq \overline{y}'$. Our results would be unchanged.

$P_U(\mathbf{y}', \mathbf{u}) \leq P_U(\mathbf{y}, \mathbf{u})$. If, in addition, $u_\ell(y'_\ell, \overline{y}') > u_\ell(y_\ell, \overline{y})$ for some $\ell \in Q(\mathbf{y}, \mathbf{u})$, then $P_U(\mathbf{y}', \mathbf{u}) < P_U(\mathbf{y}, \mathbf{u})$.

Admittedly, *Weak Pareto* is not a fully satisfactory weakening because this axiom remains silent on *some* pairs for which one distribution is unanimously preferred and does not have more individuals who are income poor than the other distribution.[33] In spite of this shortcoming, we deem *Weak Pareto* fit for our purpose. First, *Weak Pareto* is logically stronger than the welfare-consistency requirement used by Ravallion and Chen (2011). These authors note that the poverty index must be reduced when all incomes grow by the same proportion. Indeed, equi-proportionate growth makes every individual strictly better off because own income increases, while relative income is unchanged. They encapsulate this requirement in a weak relativity axiom (WRA). As we show in online Appendix S8, the WRA is itself a weakening of *Weak Pareto*. As a result, the WRA also remains silent on the pairs for which *Weak Pareto* remains silent. Second, *Weak Pareto* is compatible with *Domination* and it retains enough 'bite' in order to fully characterize the trade-offs made by the measure.

5 Results under Heterogeneous Preferences

5.1 The Case for Hierarchical Indices

In this section, we show that when *some* utility function does not depend on relative income *below the subsistence income*, the poverty index must be hierarchical. That is, the index must attribute a larger poverty score to individuals who are absolutely income poor than to individuals who are only-relatively income poor. Formally, we denote by $U^* \subset U^B$ the subset of utility functions that are independent of relative income below the subsistence income; that is, for all $u \in U^*$, we have $u(y, \overline{y}) = u(y, \overline{y}')$ for all $y \in [0, z_a)$ and $\overline{y}, \overline{y}' \geq 0$. For instance, self-centered preferences belong to U^*.

We consider the subsets of utility functions U that contain at least one member of U^*. Those subsets U are such that $X_A \subseteq X_Q(U)$, because all individuals with a utility function in U^* prefer the reference bundle over any bundle with income

[33] Indeed, *Weak Pareto* remains silent as soon as the unanimously preferred distribution has a individual who is not welfare poor, not income poor and whose income does not grow as fast as the median income. Although *Weak Pareto* remains silent for some pairs on which the poverty comparison should be unambiguous, the poverty measures that *Weak Pareto* and *Domination* characterize do make the affirmative comparisons that we would like to impose for such pairs.

below the subsistence income. This implies that the global line is never smaller than the subsistence income.

On these subsets, Proposition 4 shows that indices satisfying *Domination* and *Weak Pareto* must belong to the family of 'hierarchical' indices (Decerf, 2021).

Definition 4 (Hierarchical poverty index)**.** *Given any $U \subseteq U^B$, we say that the additive index $P_U : \mathcal{X}_U \to \mathbb{R}$ is a hierarchical poverty index if P_U is a fair additive index with global line z such that $X_z = X_Q(U)$ and for which (i) p is strictly decreasing in its first argument on X_z and (ii) p is constant in its second argument on $X_A \cap X_z$.*

Hierarchical indices are based on a 'maximal' global line, defined by $X_z = X_Q(U)$. This means that a bundle provides income poverty status when there is *some* preference in U for which the reference bundle is preferred to that bundle.

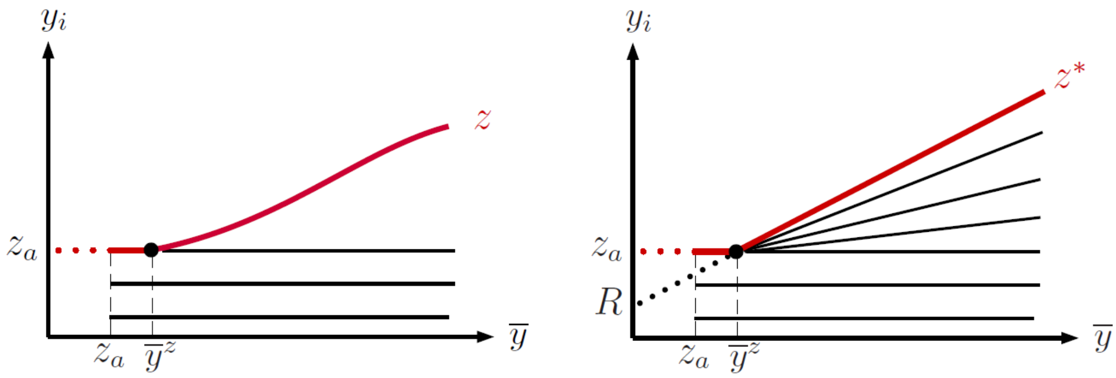
The trade-offs between own income and relative income associated with hierarchical indices are illustrated in Figure 5.a. Crucially, all iso-poverty-score curves below the subsistence income are flat lines. As a result, hierarchical indices systematically attribute a larger poverty score to an individual who is absolutely income poor than to an individual who is only-relatively income poor, regardless of the median income in their respective societies. Moreover, when comparing two absolutely income poor individuals, hierarchical indices attribute a larger poverty score to the one who earns the smaller income. The indices standardly used in the global poverty literature are not hierarchical (Decerf, 2017).

Proposition 4. *Given any $U \subseteq U^B$ with $U \cap U^* \neq \emptyset$, the additive index P_U satisfies Domination and Weak Pareto only if P_U is a hierarchical index.*

Proof. See Appendix A4. ■

The intuitive explanation for Proposition 4 is as follows. *Weak Pareto* requires that some unanimous improvements (weakly) reduce poverty. If, in addition, the utility of a welfare poor individual increases, poverty must strictly decrease. For simplicity, assume that there is only one welfare poor individual and that all other individuals have a poverty score equal to zero. Under this assumption, the poverty index must be reduced when the welfare poor individual moves to a bundle she prefers, at least if no-one is worse off in the new distribution.[34] Such

[34] The other precondition for *Weak Pareto* is that none of the other individuals becomes income poor in the new distribution. It is always possible to find an appropriate distribution-profile pair such that this other precondition is met, as we show in Lemma 3 in Appendix A1.



(a) Hierarchical index ($U \cap U^* \neq \emptyset$)

(b) Theorem 1 ($U^{\bar{\sigma}}$)

Figure 5: Iso-poverty-score curves under heterogeneous preferences.
Notes: Plain black curves are iso-poverty-score curves. The red curve is the global line.

unanimous improvement requires that the median income in the new distribution is not smaller than the median income in the initial distribution (Lemma 1). Thus, when the welfare poor individual prefers a bundle *with a larger median income*, the poverty score attributed to this bundle must be strictly smaller (Lemma 3 in Appendix A1). Importantly, this reasoning holds true for all the preferences in U that the welfare poor individual may hold. Graphically, at any bundle, the slope of the iso-poverty-score curve cannot be steeper than the slope of the indifference curve of an individual who is welfare poor at this bundle. This implies that the iso-poverty-score curves are flat below the subsistence income. Indeed, the indifference curves of a self-centered individual are flat, and moreover, said self-centered individual is welfare poor when she is absolutely income poor.[35] Thus, the iso-poverty-score curves must also be flat below z_a, because *Domination* prevents iso-poverty-score curves from having negative slopes.

Observe that the above reasoning does *not* necessarily imply that iso-poverty-score curves above z_a are flat. Even if some individuals have indifference curves that are flat above z_a, these individuals need not be welfare poor. For instance, self-centered individuals are not welfare poor above z_a.

[35]There need not be self-centered preferences in the set U, but the same reasoning holds for preferences in the set U^*.

5.2 Characterization on Set $U^{\bar{\sigma}}$

We characterize the trade-offs made by poverty indices satisfying *Domination* and *Weak Pareto* under a particular subset of U^B. Let $U^{\bar{\sigma}}$ be the subset of utility functions u^σ defined by Equation (1) for which $0 \leq \sigma < \bar{\sigma}$ for some $\bar{\sigma} > 0$. $U^{\bar{\sigma}}$ is much smaller than U^B, but $U^{\bar{\sigma}}$ features an upper bound on the sensitivity to relative income σ.

Theorem 1 shows that these measures have two key properties. First, the global line is a societal line defined as the upper contour of the subsistence income z_a and a *weakly* relative line (Ravallion and Chen, 2011) passing through the reference bundle. This global line is illustrated in Figure 5.b. On the heterogeneous set $U^{\bar{\sigma}}$, the global line yields an *imperfect identification* of the welfare poor. Although all individuals who are not income poor are not welfare poor, some income poor individuals are *not* welfare poor.

Second, the trade-offs that the indices must make between own income and relative income are completely characterized. These trade-offs are graphically illustrated in Figure 5.b, which shows their common iso-poverty-score map.[36] Below the subsistence income, the poverty score function does not depend on relative income, and thus iso-poverty-score curves are flat lines. Above the subsistence income, the poverty score function does depend on relative income. The iso-poverty-score curves are straight lines pointing to the reference bundle.

Theorem 1. *The additive index $P_{U^{\bar{\sigma}}}$ satisfies Domination and Weak Pareto if and only if $P_{U^{\bar{\sigma}}}$ is a hierarchical index with global line z^* defined as*

$$z^*(\overline{y}) := \max(z_a, R + R\bar{\sigma}\overline{y})$$

for all $\overline{y} \geq z_a$, where $R := \frac{z_a}{1+\bar{\sigma}\overline{y}^z}$, and whose poverty score function p is such that for all $(y, \overline{y}), (y', \overline{y}') \in X_{z^*} \setminus X_A$[37]

$$p(y, \overline{y}) = p(y', \overline{y}') \qquad \textit{when} \qquad \frac{y - z_a}{\overline{y} - \overline{y}^z} = \frac{y' - z_a}{\overline{y}' - \overline{y}^z}.$$

Proof. See Appendix A5. ■

[36] The indices characterized in Theorem 1 are non-continuous at the reference bundle $(z_a, \overline{y}^z)$. Recall that the poverty score function of additive indices is required to be continuous only on $X_Q(u)$, which never includes $(z_a, \overline{y}^z)$. If the poverty score function is required to be continuous on the whole domain X, then the two axioms are incompatible.

[37] A hierarchical index has for all $(y, \overline{y}), (y', \overline{y}') \in X_A$ that $p(y, \overline{y}) = p(y', \overline{y}')$ when $y = y'$.

We emphasize two interesting features of Theorem 1 that relate to the global line. First, our axioms imply the use of a *strongly* relative global line only when the sensitivity to relative income has no upper bound. Indeed, when $\bar{\sigma} \to \infty$, we have $R \to 0$ and $R\bar{\sigma} \to z_a/\overline{y}^z$, and thus the relative part of z^* tends to a strongly relative line, which points to the origin. Hence, on $U^{\bar{\sigma}}$, the global line is weakly relative when this sensitivity is bounded above ($\bar{\sigma} \in \mathbb{R}_{++}$), but strongly relative when this sensitivity has no upper bound ($\bar{\sigma} \to \infty$). Second, under heterogeneous preferences, the global line need not correspond to an indifference curve. Indeed, there is no preference in $U^{\bar{\sigma}}$ that has an indifference curve corresponding to the global line z^*. This means that there is no individual for whom the global line provides the same utility in (all) different societies. Instead, the global line defines an 'objective' income poverty status, which is attached to any individual whose income is smaller than the global line.

Observe that the trade-offs between own income and relative income made by index HH_S under the global line z^* correspond to those characterized in Theorem 1. Above the subsistence income, the poverty score function of index HH_S has the same iso-poverty-score curves.[38] Below the subsistence income, the poverty score function of index HH_S does not depend on relative income.[39]

Interestingly, Decerf (2017) proposes a family of hierarchical indices whose iso-poverty-score curves satisfy the constraints derived in Theorem 1. For a given global line z, index P^H is defined by the following poverty score function:

$$p^H_{\alpha\lambda}(y_i, \overline{y}) := \begin{cases} \left(1 - \lambda \frac{y_i}{z_a}\right)^{\alpha} & \text{if } \ y_i < z_a, \\ \left((1-\lambda) - (1-\lambda)\frac{y_i - z_a}{z(\overline{y}) - z_a}\right)^{\alpha} & \text{if } \ z_a \leq y_i < z(\overline{y}), \end{cases} \tag{9}$$

where $\alpha > 0$ and $\lambda \in (0,1)$. The hierarchical headcount ratio corresponds to the index that would be obtained when setting $\alpha = 1$ and $\lambda = 0$, the latter value being on the frontier of acceptable values.

5.3 Robustness

The trade-offs characterized in Theorem 1 are valid under subset $U^{\bar{\sigma}}$. In general, the trade-offs characterized by *Domination* and *Weak Pareto* may depend on the set U considered. We show that some of their key characteristics are preserved on

[38] On $X_{z^*} \setminus X_A$, we have that $\frac{y - z_a}{\overline{y} - \overline{y}^z} = \frac{y' - z_a}{\overline{y}' - \overline{y}^z} \Leftrightarrow \frac{z^*(\overline{y}) - y}{z^*(\overline{y}) - z_a} = \frac{z^*(\overline{y}') - y'}{z^*(\overline{y}') - z_a}$.

[39] Below z_a, HH_S attributes a poverty score equal to one regardless of own income, which is thus not strictly decreasing in own income as required by Theorem 1.

larger sets of preferences. For robustness, we consider the whole set U^B.

Theorem 2. *The additive index P_{U^B} satisfies Domination and Weak Pareto if and only if P_{U^B} is a hierarchical index with global line z^{**} defined as*

$$z^{**}(\overline{y}) := \max\left(z_a, \frac{z_a}{\overline{y}^z}\overline{y}\right)$$

for all $\overline{y} \geq z_a$, whose poverty score function p is such that for all $(y, \overline{y}), (y', \overline{y}') \in X_{z^{**}} \backslash X_A$,

$$p(y, \overline{y}) = p(y', \overline{y}') \qquad \textit{when} \qquad y = y'.$$

Proof. See online Appendix S9. ■

Interestingly, Theorem 2 shows that the iso-poverty-score curves and the relative part of the global line must be straight lines even when the set U^B contains preferences whose indifference curves are *not* straight lines.[40]

There are two key differences between Theorems 1 and 2. First, the global line of the latter is z^{**} instead of z^*. This reflects the fact that there is no upper bound on the sensitivity to relative income in U^B. Second, below the global line, the poverty score function in Theorem 2 only depends on own income. This does not mean that relative income plays no role, but its role is limited to the definition of the global line. This second difference reflects the fact that U^B contains utility functions that exhibit extreme forms of concavities. This difference disappears on the subset U^C of utility functions in U^B whose indifference curves are weakly convex in the space of bundles. On U^C, the iso-poverty-score map characterized by *Domination* and *Weak Pareto* corresponds to that illustrated in Figure 5.b, except that the global line is then z^{**} because no upper bound is placed on the sensitivity to relative income in U^C.[41]

We emphasize that our results also hold when assuming that utility is *strictly* increasing in relative income, instead of *weakly* increasing. Indeed, Theorem 1 would be unchanged if the subset $U^{\bar{\sigma}}$ excludes u^0.[42]

[40] For the relative part of the global line, the explanation follows from the fact that the global line is maximal. When $\overline{y} \geq \overline{y}^z$, the shape of the global line is defined by the utility function most sensitive to relative income. On U^B, the strict monotonicity to own income when holding relative income constant constrains the sensitivity to relative income. Graphically, the slope of the indifference curve passing through a bundle cannot exceed the slope of the ray passing through this bundle and the origin.

[41] Details can be found in online Appendix S10.

[42] Details can be found in online Appendix S11.

6 Empirical Application

In this section we apply our proposed index, the hierarchical headcount ratio (HH_S), to measure global poverty. The goal of this exercise is to illustrate how the switch of index can affect the assessment of poverty. We compare the evaluation of poverty according to the hierarchical headcount ratio both with the societal headcount ratio (H_S) using the same global poverty line, and with the absolute headcount ratio (H_A), arguably the most well-known poverty measure. We compare poverty at the country level as well as the evolution and distribution of global poverty. Besides the theoretical arguments for moving away from the societal headcount ratio, ultimately, such a change of index would be justified only if it significantly affects the measurement of global poverty.

6.1 Data and Parameters

Our source of data is PovcalNet,[43] the on-line tool for poverty measurement developed by the World Bank, which offers income or consumption data from more than 1500 household surveys across more than 160 countries. In order to allow for cross-country comparisons, the World Bank converts income or consumption using the 2011 PPP exchange rates for household consumption from the International Comparison Program. Moreover, for the purpose of performing multi-country aggregations the World Bank defines reference years aligning survey estimates from different years. To analyze poverty at the global or regional level, we only use reference years.[44] We use data from 1999 up to 2015, the most recent reference year available. We take 1999 as our base year because before that, we have $HH_S = H_S$ in several populous countries that still had median income below $\overline{y}^z$. Our dataset includes 168 countries.[45]

To estimate the hierarchical headcount ratio, we need to choose a societal poverty line. In our theoretical framework, two exogenous elements determine the societal poverty line: the reference bundle $(z_a, \overline{y}^z)$ and the set of admissible (ordinal) utility functions U representing preferences over own and relative income. The specification of the reference bundle is clearly a normative choice,

[43]PovcalNet can be found at: http://iresearch.worldbank.org/PovcalNet/povOnDemand.aspx. The data was retrieved in April 2021. All data is in per-capita terms.

[44]The reference years available from 1999 on are: 1999, 2002, 2005, 2008, 2010-2013 and 2015.

[45]China, India, and Indonesia compile data for rural and urban areas separately. Thus, we have 171 units in total. A few countries do not have data for all reference years. These countries are: Kosovo (missing data in 1999), Nauru (missing data before 2005), Somalia (missing data before 2011), South Sudan (missing data before 2008), Timor-Leste (missing data in 1999), and Venezuela (missing data in 2015). Our results are robust to excluding this set of countries.

while the identification of U is more subtle. It could be taken as the set of all preferences actually held by some individual, which is, in principle, empirically observable. Under this approach, identification of U is a positive exercise. It could, alternatively, be taken as another normative choice to be made by the social planner. The social planner might make some normative evaluation of 'reasonableness' of preferences and only include in U preferences that she considers to be 'reasonable'. The choice of approach is of course, itself, a normative choice.

For the purposes of our empirical analysis, we infer both the reference bundle $(z_a, \overline{y}^z)$ and the set of admissible utility functions U from the societal poverty line currently used by the World Bank to assess income poverty, combining its absolute and relative aspects. In doing so, we elucidate the normative choices implicit in the World Bank approach from the perspective of our theoretical framework. This has the additional advantage that the societal headcount ratio (H_S) corresponds exactly to the World Bank's official societal poverty measure, while the absolute headcount ratio (H_A) corresponds to its official extreme poverty measure, thus facilitating comparison of existing global poverty assessments with those of our proposed measure.

The World Bank's societal poverty line, $max(\$1.90, \$1.00 + 0.5\overline{y})$ where $\overline{y}$ is country median income, is the upper contour of an absolute poverty line and a weakly relative line (World Bank, 2018). The absolute line of \$1.90 per person per day, in 2011 PPP, has been the official World Bank extreme poverty line since 2015 (Ferreira et al., 2016). The weakly relative line $\$1.00 + 0.5\overline{y}$ was estimated from regressions of 699 (national) poverty lines against median income (Jolliffe and Prydz, 2021).

The components of the reference bundle are thus straightforwardly inferred from the intersection of the absolute and weakly relative lines as subsistence income $z_a = \$1.90$ and reference median income $\overline{y}^z = \$1.80$, the latter being the maximal median income at which the subsistence income is considered sufficient for social inclusion. We observe with interest that the reference median income is close to, but slightly less than, the subsistence income. It follows that an individual whose income is z_a, who lives in a country where the median income is also equal to z_a, is not considered to have sufficient income for social inclusion. Meanwhile, the set of admissible preferences U corresponds to members of the parametric family specified by Equation (1) with parameter values σ in the interval $[0, 0.5]$. The extent to which this set aligns with the actual heterogeneity of preferences in the global population remains an open empirical question, beyond the scope of the present study.

6.2 Empirical Values for the Weight of HH_S

We first provide some empirical insights on the hierarchical headcount ratio (HH_S). As seen in Equation (5), the hierarchical headcount ratio can be decomposed as the sum of the fraction of individuals who are absolutely income poor (H_A) and the fraction of individuals who are only-relatively income poor (H_R) multiplied by a weight $\omega(\mathbf{y}) \in [0, 1]$.

Table 1 displays the median weight by country income group as defined by the World Bank. The results highlight two main points. First, we observe that for low- and middle-income countries, the median weight is close to 0.5 and there is little variation within each group, especially in the low-income and lower-middle-income groups. This means that the median poverty score of the only-relatively income poor individuals in low- and middle-income countries is approximately 0.5. More specifically, we observe that more than 60% of countries in our sample have a median endogenous weight between 0.4 and 0.55. Except in two cases, all these are low- or middle-income countries (see also Figure S.1 in online Appendix S4). This implies that for many countries in our sample, HH_S and the simpler index $H_S^{1/2}$ would yield similar poverty evaluations. Recall that $H_S^{1/2}$ is defined similarly to HH_S but with a fixed weight of 0.5 (see Section 2).

The fact that $\omega(\mathbf{y})$ is close to 0.5 for many low- and middle-income countries can be explained intuitively. In those countries, the relative line is close to the absolute line. As a result, the density function of the income distribution is close to being constant in the small interval between lines. In other words, the distribution of individual incomes between z_a and $z(\overline{y})$ is close to being uniform. In such a case, the average income for the only-relatively income poor is $\hat{y}^R \approx \frac{z_a + z(\overline{y})}{2}$, which yields $\omega(\mathbf{y}) \approx 0.5$ when $z_a \approx z(\overline{y})$.

Second, we observe that the weight decreases as income grows along country groups. Indeed, the median weight among high-income countries is 0.31, considerably smaller than in the other groups. This implies that the incomes of the only-relatively income poor in richer countries are, on average, proportionally closer to the relative line than to the absolute line than the incomes of the only-relatively income poor in low-income countries.

The results summarized in Table 1 help us to understand why, when we compute global poverty (see Section 6.4), HH_S is about halfway between H_S and H_A. The reason is that income poverty is concentrated in highly populated developing countries such as India and Indonesia, where the endogenous weight is close to 0.5.

Table 1: Endogenous weight ω by country income group

Country income group	Median	SD
Low income	0.49	0.06
Lower-middle income	0.48	0.05
Upper-middle income	0.42	0.08
High income	0.31	0.06

Note: Country income groups as defined by the World Bank.

6.3 Country-Level Poverty

We begin by comparing poverty at the country level between the societal headcount ratio (H_S), the absolute headcount ratio (H_A), and our proposed index (HH_S).

Our first example illustrates how the change of index that we propose can affect the poverty diagnoses of individual countries. Table 2 displays the level of poverty as well as the rankings by those three poverty measures for Botswana and Egypt in 2008. According to H_S, Egypt has more poverty than Botswana (the fraction of income poor individuals is 33% and 30%, respectively). However, this comparison masks the very different composition of their income poor populations. The vast majority of income poor individuals in Egypt are only-relatively income poor, whereas about half of the income poor individuals in Botswana are absolutely income poor. Specifically, the fractions of absolutely and only-relatively income poor individuals are, respectively, 5% and 28% in Egypt, and 16% and 14% in Botswana. Similarly to H_A, Botswana has more poverty than Egypt by HH_S. The reason for these opposite diagnoses between H_S and HH_S is that while H_S gives the same poverty score to all income poor individuals, HH_S down-weights individuals who are only-relatively income poor to around 0.5. As explained in Section 2, a weight smaller than one reflects the fact that not all only-relatively income poor individuals are welfare poor.

These differences imply that the country-rankings of Egypt and Botswana differ considerably between H_S and HH_S. While Egypt is 11 places below Botswana by H_S, it is 18 places higher by HH_S. More generally, we could also estimate how many re-rankings might occur overall, on average, for all countries. When considering all countries and all years in the sample, we observe an average absolute difference in rankings between H_S and HH_S of 3.

Our next example illustrates how trends can diverge across the three poverty measures. Figure 6 displays the evolution of poverty in Samoa over 2002-2008,

Table 2: Egypt and Botswana: Poverty values and rankings in 2008

	Values			Rankings			$\omega(\mathbf{y})$	Median inc.
	H_S	HH_S	H_A	H_S	HH_S	H_A		(PPP$)
Botswana	29.7	22.9	15.8	89	104	112	0.51	117
Egypt	32.9	17.1	4.7	100	86	83	0.44	132

as captured by H_A, H_S, and HH_S. Samoa experienced unequal income growth over 2002-2008; that is, both the standards of living (measured by either mean or median income) and inequality (measured by the Gini index) increased. The greater inequality resulted in the fraction of individuals who were income poor, as measured by H_S, increasing by 7% over this period. However, the economic growth meant that over time, a considerably smaller fraction of those income poor remained absolutely income poor. In fact, the absolute headcount ratio decreased by 70% over this period. Thus, many absolutely income poor individuals became only-relatively income poor, an evolution not captured by H_S. HH_S, on the other hand, does capture the shift from absolute to only-relative income poverty, decreasing by 9% over 2002-2008. Moving beyond this illustrative example, we can estimate how often these opposite conclusions between H_S and HH_S occur in our sample. Considering only countries with $z_a < z(\overline{y})$, for which H_S and HH_S do not coincide, we compare the evolution of H_S and HH_S over all $t/t-4$ periods for all countries in the sample.[46] The results show that the share of opposite trends between H_S and HH_S is 9.7%.

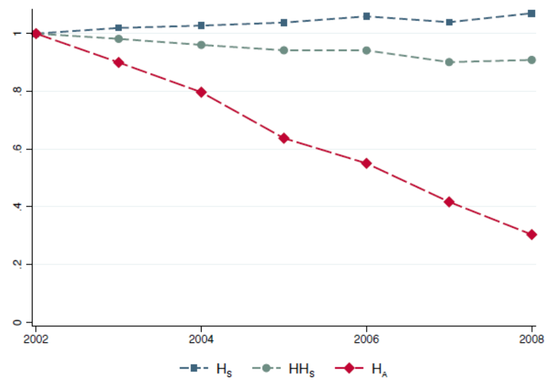


Figure 6: Evolution of poverty in Samoa, 2002-2008.

Note: The graph plots the evolution of poverty relative to 2002 for all years through 2008.

[46] We use the following reference years: 1999, 2003, 2007, 2011, and 2015.

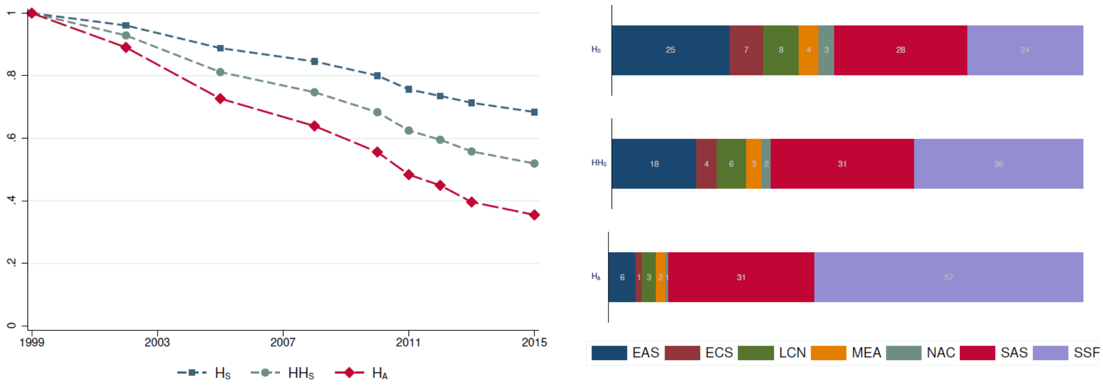
6.4 Global Poverty: Trends and Distribution

We turn now to studying how global poverty varies between the absolute headcount ratio (H_A), the societal headcount ratio (H_S), and the hierarchical headcount ratio (HH_S). First, we analyze the trends in global poverty according to these three measures. Figure 7a shows that HH_S decreases by 48% between 1999 and 2015, while H_S decreases by 'only' 32% over the same period (see also Table S.1 in online Appendix S4). This result, albeit meaningful, is not surprising given the definitions of H_S and HH_S, and the values of ω that we observe in the data. The main reason for this difference is that the sharp decrease in H_A over 1999-2015, which amounts to 64%, has a stronger effect on HH_S than on H_S. Indeed, HH_S systematically decreases when an individual leaves absolute income poverty, while H_S is unchanged if the individual becomes only-relatively income poor.

Turning the focus to regional poverty, we show that and how the proposed adoption of HH_S affects the distribution of global poverty across regions. Figure 7b displays the regional distribution of global poverty in 2015 for the three measures. We highlight two major differences in the current distribution of poverty across regions. First, the share of global poverty in East Asia and the Pacific is considerably smaller for HH_S than for H_S. In 2015, this region accounted for 18% of global poverty according to HH_S while accounting for 25% according to H_S. This difference in the weight of East Asia and the Pacific is to a large extent explained by the distribution of absolute income poverty. Indeed, in 2015, only 6% of the world's population suffering from absolute income poverty lived in East Asia and the Pacific.

Second, Sub-Saharan Africa has a considerably larger share of global poverty according to HH_S than according to H_S. In 2015, this region accounted for the largest share of global poverty according to HH_S, amounting to 36%, while being the third region according to H_S, at 24%. Again, this increase in the share of global poverty in Sub-Saharan African when H_S is replaced by HH_S is to a large extent explained by the fact that this region was host to almost 60% of absolutely income poor individuals.

As discussed in Section 6.2 and easily seen in Figure 7a, at the worldwide level, HH_S lies very much in between H_S and H_A because many low- and middle-income countries have a $\omega \approx 0.5$. Hence, one could wonder whether we could obtain similar results to HH_S if we instead use the even simpler index $H_S^{1/2}$. As expected, given our previous analysis, Figures S.3a and S.3b in online Appendix S4 show that using $H_S^{1/2}$ yields very similar results as HH_S when measuring both



(a) Evolution of global poverty, 1999-2015 **(b)** Distribution by region, 2015

Figure 7: Evolution and distribution of global poverty

Notes: Panel a plots the evolution of poverty relative to 1999 for all reference years through 2015. It includes all countries with available information in each reference year. See Footnote 45 for a list of countries with missing information in given reference years. Panel b plots the contribution to global poverty for each of the following regions: East Asia & Pacific (EAS), Europe & Central Asia (ECS), Latin America & Caribbean (LCN), Middle East & North Africa (MEA), North America (NAC), South Asia (SAS), and Sub-Saharan Africa (SSF).

the evolution of global poverty and its distribution across regions, respectively. We can then safely conclude that if HH_S were to be deemed too complex to implement and $H_S^{1/2}$ used instead, it would make little difference in terms of the revised poverty diagnoses for both many low- and middle-income countries (for which $\omega \approx 0.5$) and the world as a whole (at least in the near future).

7 Concluding Remarks

We have developed a theory of global income poverty measurement with preference heterogeneity over own income and relative income. This theory fills three gaps in the literature. First, it shows how the poverty measure can aggregate heterogeneous individual preferences. Second, it provides a welfarist foundation for the societal lines proposed in the literature. Third, it shows that the standard indices such as the societal headcount ratio violate the most basic welfare-consistency property when preferences are heterogeneous. We show that a simple modification of the societal headcount is better aligned with the theory, and that this proposed switch yields a different picture of the evolution and distribution of income poverty both at the country and world level.

Our paper leaves several questions unanswered and additional research is called for. First, the reference bundle is taken to be exogenously chosen by some social planner. Although such exogeneity is sufficient to provide the foundations for global lines of the ‘societal’ type, it implies that our theory cannot pin down the exact design of the global line. Thus, the construction of a fair and welfare-consistent global poverty measure requires bringing together the insights of our results with those of the literature on global lines (Ravallion, 2020). Furthermore, we demonstrated that the design of the global line depends on the set of admissible preferences as well as the specification of the reference bundle. While this set of preferences could also be taken as an exogenous choice of the social planner, it could alternatively represent the empirical diversity of preferences over own and relative income in the global population. We leave elicitation of such preferences for future research. Second, our results rule out the use of standard poverty indices in our framework, but do not pin down a unique index to replace them. We argue that the hierarchical headcount ratio would be a natural candidate to replace the societal headcount ratio because it strikes a good compromise between simplicity and having desirable properties. However, this is a mere proposal and the selection of the ideal index for our framework remains an open question.

Finally, we motivate our paper based on *global* income poverty measurement. More generally, our theory would be relevant for any government or regional entity concerned with both subsistence and social inclusion, such as, for example, the European Union (European Commission, 2015).

References

Alkire, S. and Foster, J. (2011). Counting and Multidimensional Poverty Measurement. *Journal of Public Economics*, 95(7-8):476–487.

Andreoni, J. and Vesterlund, L. (2001). Which Is the Fair Sex? Gender Differences in Altruism. *The Quarterly Journal of Economics*, 116(1):293–312.

Atkin, D. (2013). Trade, tastes, and nutrition in India. *American Economic Review*, 103(5):1629–63.

Atkin, D. (2016). The caloric costs of culture: Evidence from Indian migrants. *American Economic Review*, 106(4):1144–81.

Atkinson, A. (2016). *Monitoring Global Poverty: Report of the Commission on Global Poverty.* The World Bank, Washigton, DC.

Atkinson, A. and Bourguignon, F. (2001). Poverty and Inclusion from a World Perspective. In Stiglitz, J. and Muet, P.-A., editors, *Governance, Equity and Global Markets*. Oxford University Press, New York.

Blanco, M., Engelmann, D., and Theo, H. (2011). A Within-Subject Analysis of Other-Regarding Preferences. *Games and Economic Behavior*, 72(2):321–338.

Brun, B. C. and Tungodden, B. (2004). Non-welfaristic theories of justice: Is "the intersection approach" a solution to the indexing impasse? *Social Choice and Welfare*, 22(1):49–60.

Clark, A. E. and Oswald, A. J. (1996). Satisfaction and Comparison Income. *Journal of Public Economics*, 61(3):359–381.

Decancq, K., Fleurbaey, M., and Maniquet, F. (2019). Multidimensional Poverty Measurement with Individual Preferences. *Journal of Economic Inequality*, 17(1):29–49.

Decancq, K., Fleurbaey, M., and Schokkaert, E. (2015). Happiness, Equivalent Incomes and Respect for Individual Preferences. *Economica*, 82:1082–1106.

Decerf, B. (2017). Why Not Consider That Being Absolutely Poor Is Worse Than Being Only Relatively Poor? *Journal of Public Economics*, 152:79–92.

Decerf, B. (2021). Combining absolute and relative poverty: Income poverty measurement with two poverty lines. *Social Choice and Welfare*, 56(2):325–362.

Decerf, B. and Ferrando, M. (2022). Unambiguous trends combining absolute and relative income poverty: New results and global application. *The World Bank Economic Review*, 36(3):605–628.

Dimri, A. and Maniquet, F. (2020). Income poverty measurement in India: Defining group-specific poverty lines or taking preferences into account? *The Journal of Economic Inequality*, 18(2):137–156.

Eckel, C. and Grossman, P. (1998). Are women less selfish than men?: Evidence from dictator experiments. *The Economic Journal*, 108(448):726–735.

European Commission (2015). *Portfolio of Indicators for the Monitoring of the European Strategy for Social Protection and Social Inclusion.* European Commission, Brussels.

Ferreira, F. H., Chen, S., Dabalen, A., Dikhanov, Y., Hamadeh, N., Jolliffe, D., Narayan, A., Prydz, E. B., Revenga, A., Sangraula, P., Serajuddin, U., and Yoshida, N. (2016). A global count of the extreme poor in 2012: data issues, methodology and initial results. *Journal of Economic Inequality*, 14(2):141–172.

Fleurbaey, M. and Maniquet, F. (2006). Fair income tax. *The Review of Economic Studies*, 73(1):55–83.

Fleurbaey, M. and Maniquet, F. (2011). *A Theory of Fairness and Social Welfare.* Cambridge University Press, Cambridge.

Fleurbaey, M. and Trannoy, A. (2003). The impossibility of a Paretian egalitarian. *Social Choice and Welfare*, 21(2):243–263.

Foster, J. and Shorrocks, A. (1991). Subgroup Consistent Poverty Indices. *Econometrica*, 59(3):687–709.

Jolliffe, D. and Prydz, E. B. (2021). Societal poverty: A relative and relevant measure. *The World Bank Economic Review*, 35(1):180–206.

Luttmer, E. F. (2005). Neighbors as Negatices : Relative Earnings and Well-Being. *Quaterly Journal of Economics*, 120(3):963–1002.

Perez-Truglia, R. (2020). The Effects of Income Transparency on Well-Being : Evidence from a Natural Experiment. *American Economic Review*, 110(4):1019–1054.

Ravallion, M. (1998). *Poverty Lines in Theory and Practice.* The World Bank.

Ravallion, M. (2020). On measuring global poverty. *Annual Review of Economics*, 12(1):167–188.

Ravallion, M. and Chen, S. (2011). Weakly relative poverty. *Review of Economics and Statistics*, 93(4):1251–1261.

Ravallion, M. and Chen, S. (2019). Global poverty measurement when relative income matters. *Journal of Public Economics*, 177:104046.

Sen, A. (1976). Poverty: an Ordinal Approach to Measurement. *Econometrica*, 44(2):219–231.

Sen, A. (1985). Commodities and capabilities. In *Professor Dr. P. Hennipman Lectures in Economics: Theory, Institutions, Policy*, volume 7. Elsevier, Amsterdam.

Smith, A. (1776). An inquiry into the nature and causes of the wealth of nations: Volume one. London: printed for W. Strahan; and T. Cadell, 1776.

Sprumont, Y. (2012). Resource egalitarianism with a dash of efficiency. *Journal of Economic Theory*, 147(4):1602–1613.

Townsend, P. (1985). A Sociological Approach to the Measurement of Poverty - A Rejoinder to Professor Amartya Sen. *Oxford Economic Papers*, 37(4):659–668.

Treibich, R. (2019). Welfare egalitarianism with other-regarding preferences. *Social Choice and Welfare*, 52(1):1–28.

Van Veelen, M. and van der Weide, R. (2008). A note on different approaches to index number theory. *American Economic Review*, 98(4):1722–30.

World Bank (2018). *Poverty and Shared Prosperity 2018: Piecing Together the Poverty Puzzle.* World Bank, Washington, DC.

Appendix

A1 Technical Lemmas

Lemma 2. *Given any $U \subseteq U^B$, for all $(y, \overline{y}) \in X$ with $y \geq \overline{y}$, we have $(y, \overline{y}) \notin X_Q(U)$.*

Proof. We have $u(z_a, z_a) \geq u(z_a, \overline{y}^z)$ for all $u \in U$ because utility functions are weakly increasing in relative income when holding own income constant and because we assume $\overline{y}^z \geq z_a$. We have $u(\overline{y}, \overline{y}) \geq u(z_a, z_a)$ for all $u \in U$ because utility functions are increasing in own income when holding relative income constant and because we assume $\overline{y} \geq z_a$. By transitivity, we have $u(\overline{y}, \overline{y}) \geq u(z_a, \overline{y}^z)$. This, in turn, implies that $u(y, \overline{y}) \geq u(z_a, \overline{y}^z)$ for all $u \in U$ because $y \geq \overline{y}$. By definition this yields $(y, \overline{y}) \notin X_Q(U)$, the desired result. ∎

Lemma 3. *Consider any $U \subseteq U^B$, any fair additive index P_U satisfying Weak Pareto and any two bundles $(y, \overline{y}), (y', \overline{y}') \in X$ with $\overline{y} \leq \overline{y}'$. If there exists some $u \in U$ such that $(y, \overline{y}) \in X_Q(u)$ and $u(y', \overline{y}') \geq u(y, \overline{y})$, then $p(y', \overline{y}') \leq p(y, \overline{y})$. If, in addition, $u(y', \overline{y}') > u(y, \overline{y})$, then $p(y', \overline{y}') < p(y, \overline{y})$.*

Proof. Consider the two distributions $\mathbf{y} := (y, \overline{y}, \overline{y})$ and $\mathbf{y}' := (y', \overline{y}', \overline{y}')$, for which $\overline{\mathbf{y}} = \overline{y}$ and $\overline{\mathbf{y}}' = \overline{y}'$. For some $u' \in U$, consider the preferences profile $\mathbf{u} := (u, u', u') \in U^3$. By construction, we have $(y_1, \overline{y}) = (y, \overline{y})$, $(y'_1, \overline{y}') = (y', \overline{y}')$ and $u_1 = u$. Also, we have $(y_i, \overline{y}) = (\overline{y}, \overline{y})$, $(y'_i, \overline{y}') = (\overline{y}', \overline{y}')$ and $u_i = u'$ for all $i \in \{2, 3\}$.

First, we show that $p(y_i, \overline{y}) = p(y'_i, \overline{y}') = 0$ for all $i \in \{2, 3\}$. We have $(y_i, \overline{y}), (y'_i, \overline{y}') \notin X_Q(U)$ for all $i \in \{2, 3\}$ since $(\overline{y}, \overline{y}), (\overline{y}', \overline{y}') \notin X_Q(U)$ (Lemma 2). This implies that $(y_i, \overline{y}), (y'_i, \overline{y}') \notin X_z$ because a fair additive index has $X_z \subseteq X_Q(U)$. In turn, the fact that $(y_i, \overline{y}), (y'_i, \overline{y}') \notin X_z$ for all $i \in \{2, 3\}$ implies $p(y_i, \overline{y}) = p(y'_i, \overline{y}') = 0$ for all $i \in \{2, 3\}$ by definition of a fair additive index.

Second, we show that $P(\mathbf{y}, \mathbf{u}) \geq P(\mathbf{y}', \mathbf{u})$. We have $1 \in Q(\mathbf{y}, \mathbf{u})$ because $(y_1, \overline{y}) \in X_Q(u_1)$ since $(y, \overline{y}) \in X_Q(u)$. We have $u_1(y_1, \overline{y}) \leq u_1(y'_1, \overline{y}')$ because $u(y, \overline{y}) \leq u(y', \overline{y}')$. Consider $\lambda := \overline{y}'/\overline{y}$, which is such that $\lambda \geq 1$. We have $(y'_i, \overline{y}') = (\lambda y_i, \lambda \overline{y})$ for all $i \in \{2, 3\}$. This implies $u_i(y_i, \overline{y}) \leq u_i(\lambda y_i, \lambda \overline{y})$ for all $i \in \{2, 3\}$ because utility is increasing in own income when relative income is kept constant. We directly get $u_i(y_i, \overline{y}) \leq u_i(y'_i, \overline{y}')$ for all $i \in \{2, 3\}$. Hence, Weak Pareto implies that $P(\mathbf{y}, \mathbf{u}) \geq P(\mathbf{y}', \mathbf{u})$.

We are now equipped to prove the statement. The fact that $P(\mathbf{y}, \mathbf{u}) \geq P(\mathbf{y}', \mathbf{u})$ and $p(y_i, \overline{y}) = p(y_i', \overline{y}') = 0$ for all $i \in \{2, 3\}$ implies that $p(y_1, \overline{y}) \geq p(y_1', \overline{y}')$. This directly yields $p(y, \overline{y}) \geq p(y', \overline{y}')$. If, in addition, $u(y, \overline{y}) < u(y', \overline{y}')$, then the same argument shows that *Weak Pareto* implies $P(\mathbf{y}, \mathbf{u}) > P(\mathbf{y}', \mathbf{u})$ and thus $p(y, \overline{y}) > p(y', \overline{y}')$, the desired result. ■

Lemma 4. *Given any $U \subseteq U^B$, any fair additive index P_U satisfying Weak Pareto has $X_z = X_Q(U)$.*

Proof. Assume to the contrary that $X_z \neq X_Q(U)$. This implies that $X_z \subset X_Q(U)$ because a fair additive index must have $X_z \subseteq X_Q(U)$. Hence, there must exist some $(y, \overline{y}) \in X_Q(U)$ with $y > 0$ for which $(y, \overline{y}) \notin X_z$. As $(y, \overline{y}) \notin X_z$, we have $p(y, \overline{y}) = 0$ because P_U is a fair additive index. As $(y, \overline{y}) \in X_Q(U)$, there exists some $u' \in U$ such that $(y, \overline{y}) \in X_Q(u')$. For any $\lambda > 1$, we have that bundle $(\lambda y, \lambda \overline{y}) \in X$ is such that $u'(\lambda y, \lambda \overline{y}) > u'(y, \overline{y})$ because utility is strictly increasing in own income when relative income is constant. As P_U satisfies *Weak Pareto*, Lemma 3 implies that $p(\lambda y, \lambda \overline{y}) < p(y, \overline{y})$. This is a contradiction to $p(y, \overline{y}) = 0$ because 0 is the minimum value that function p can take. ■

Lemma 5. *Consider any $\bar{\sigma} > 0$, any $\overline{y} > \overline{y}^z$, and any $y \in [z_a, z^*]$ where $z^* := R + R\bar{\sigma}\overline{y}$ and $R := \frac{z_a}{1+\bar{\sigma}\overline{y}^z}$. The value of sensitivity to relative income*

$$\sigma^* := \frac{y - z_a}{z_a \overline{y} - y \overline{y}^z} \tag{A.1}$$

is such that (i) $\sigma^* \in [0, \bar{\sigma}]$, (ii) $u^{\sigma^*}(y, \overline{y}) = u^{\sigma^*}(z_a, \overline{y}^z)$, and (iii) $(y, \overline{y}) \in X_Q(u^\sigma) \Leftrightarrow \sigma > \sigma^*$. Moreover, for any $y' \geq z_a$ and $\overline{y}' > \overline{y}^z$ with $\frac{y' - z_a}{\overline{y}' - \overline{y}^z} = \frac{y - z_a}{\overline{y} - \overline{y}^z}$, we have (iv) $u^{\sigma^*}(y', \overline{y}') = u^{\sigma^*}(y, \overline{y})$, and (v) if $\overline{y}' > \overline{y}$, then $u^\sigma(y', \overline{y}') < u^\sigma(y, \overline{y})$ for all $\sigma > \sigma^*$.

Proof. The proof uses the observation that u^σ is ordinally equivalent to $\hat{u}^\sigma := -(u^\sigma)^{-1}$. From Equation (1), we get $\hat{u}^\sigma(y, \overline{y}) = \frac{y}{1+\sigma\overline{y}}$.

Part (i). In order to show that $\sigma^* \geq 0$, it is sufficient to prove that its denominator is positive because by assumption $y \geq z_a$. Thus, we must show that $z_a \overline{y} > y \overline{y}^z$. By assumption, we have $y \leq z^*$, so it is sufficient to show that $z_a \overline{y} > z^* \overline{y}^z$. Replacing z^* by its expression in last inequality yields $\frac{\overline{y}}{\overline{y}^z} > \frac{1+\bar{\sigma}\overline{y}}{1+\bar{\sigma}\overline{y}^z}$. This inequality holds because $\overline{y} > \overline{y}^z$ and $\bar{\sigma} > 0$, which proves that $\sigma^* \geq 0$. In order to show that $\sigma^* \leq \bar{\sigma}$, it is sufficient to show that it holds for $y = z^*$, because $\frac{\partial \sigma^*}{\partial y} \geq 0$ and $y \leq z^*$. Replacing y by z^* in the expression for σ^* yields after some manipulations $\sigma^* = \bar{\sigma}$, the desired result.

Part (ii). Let $y^z \geq 0$ be the income level implicitly defined by

$$u^{\sigma^*}(y^z, \overline{y}^z) = u^{\sigma^*}(y, \overline{y}).$$

In words, an individual with preference u^{σ^*} is indifferent between bundle $(y, \overline{y})$ and earning y^z when median income is $\overline{y}^z$. In order to show that $u^{\sigma^*}(z_a, \overline{y}^z) = u^{\sigma^*}(y, \overline{y})$, we show that $y^z = z_a$. As u^{σ^*} is ordinally equivalent to $\hat{u}^{\sigma^*}$, the definition of y^z yields $\hat{u}^{\sigma^*}(y^z, \overline{y}^z) = \hat{u}^{\sigma^*}(y, \overline{y})$. By the definition of $\hat{u}^{\sigma}$, this yields $y^z = y\frac{1+\sigma^*\overline{y}^z}{1+\sigma^*\overline{y}}$. By replacing σ^* by its expression in that of y^z, we get after some manipulations $y^z = z_a$, as desired.
Part (iii). By definition, we have $(y, \overline{y}) \in X_Q(u^{\sigma})$ if and only if $u^{\sigma}(y, \overline{y}) < u^{\sigma}(z_a, \overline{y}^z)$. As u^{σ} is ordinally equivalent to $\hat{u}^{\sigma}$, last inequality is equivalent to $\frac{y}{1+\sigma\overline{y}} < \frac{z_a}{1+\sigma\overline{y}^z}$. A few manipulations yield $\frac{y-z_a}{z_a\overline{y}-y\overline{y}^z} < \sigma$, as desired.
Part (iv). For all $\sigma \geq 0$, let $y^{\sigma} \geq 0$ be the income level implicitly defined by

$$u^{\sigma}(y^{\sigma}, \overline{y}') = u^{\sigma}(y, \overline{y}).$$

As u^{σ} is ordinally equivalent to $\hat{u}^{\sigma}$, we get $y^{\sigma} = y\frac{1+\sigma\overline{y}'}{1+\sigma\overline{y}}$. For the case $\sigma = \sigma^*$, we have $u^{\sigma^*}(y^{\sigma^*}, \overline{y}') = u^{\sigma^*}(y, \overline{y})$ when $y^{\sigma^*} = y\frac{1+\sigma^*\overline{y}'}{1+\sigma^*\overline{y}}$. In order to show that $u^{\sigma^*}(y', \overline{y}') = u^{\sigma^*}(y, \overline{y})$, we show that $y^{\sigma^*} = y'$. By replacing σ^* by its expression in that of y^{σ^*}, we get, with a few manipulations,

$$y^{\sigma^*} = \frac{z_a(\overline{y} - \overline{y}') + y(\overline{y}' - \overline{y}^z)}{\overline{y} - \overline{y}^z}. \tag{A.2}$$

We also have $y = z_a + \frac{y'-z_a}{\overline{y}'-\overline{y}^z}(\overline{y} - \overline{y}^z)$ because $\frac{y-z_a}{\overline{y}-\overline{y}^z} = \frac{y'-z_a}{\overline{y}'-\overline{y}^z}$. By replacing y in Equation (A.2) by its expression, we get with a few manipulations, $y^{\sigma^*} = y'$, as desired.
Part (v). By the definition of y^{σ} (see proof of Part (iv)), we have $u^{\sigma}(y', \overline{y}') < u^{\sigma}(y, \overline{y})$ if and only if $y' < y^{\sigma}$, because utility is strictly increasing in own income. The proof of Part (iv) shows that $y^{\sigma^*} = y'$. Thus, we have $u^{\sigma}(y', \overline{y}') < u^{\sigma}(y, \overline{y})$ for all $\sigma > \sigma^*$ if we have $\frac{\partial y^{\sigma}}{\partial \sigma} > 0$. From the expression of y^{σ}, we get $\frac{\partial y^{\sigma}}{\partial \sigma} = y\frac{\overline{y}'-\overline{y}}{(1+\sigma\overline{y})^2}$. We therefore have $\frac{\partial y^{\sigma}}{\partial \sigma} > 0$ because $y \geq z_a$ and $\overline{y}' > \overline{y} > 0$, the desired result. ■

A2 Proof of Proposition 3

As set U is heterogeneous, there exist two $u, u' \in U$ and some $(y, \overline{y})$ with $\overline{y} \geq \overline{y}^z$ such that $(y, \overline{y}) \in X_Q(u)$ and $(y, \overline{y}) \notin X_Q(u')$. P_U is a fair additive index because

P_U satisfies *Domination* (Proposition 1). As P_U also satisfies *Pareto*, Lemma 4 implies that $X_z = X_Q(U)$.[47] Therefore, $(y, \overline{y}) \in X_z$ because $(y, \overline{y}) \in X_Q(u)$.

Consider the two distributions $\mathbf{y} := (y, \overline{y}, \overline{y})$ and $\mathbf{y}' := (z_a, \overline{y}^z, \overline{y}^z)$, for which $\overline{\mathbf{y}} = \overline{y}$ and $\overline{\mathbf{y}}' = \overline{y}^z$. By construction, we have $(y_1, \overline{y}) = (y, \overline{y})$, $(y_1', \overline{y}') = (z_a, \overline{y}^z)$. Also, we have $(y_i, \overline{y}) = (\overline{y}, \overline{y})$ and $(y_i', \overline{y}') = (\overline{y}^z, \overline{y}^z)$ for all $i \in \{2, 3\}$. We have $(\overline{y}, \overline{y}), (\overline{y}^z, \overline{y}^z) \notin X_Q(U)$ (Lemma 2). And we have by construction of $\mathbf{y}'$ that $(y_i', \overline{y}') \notin X_z$ for all $i \in \{1, 2, 3\}$ because $(z_a, \overline{y}^z) \notin X_Q(U)$ and $X_z = X_Q(U)$. Therefore, we have $P_U(\mathbf{y}', \mathbf{u}) = \hat{k}$ for all $\mathbf{u} \in U^3$ because P_U is a fair additive index (since it satisfies *Domination*). In contrast, $(y_1, \overline{y}) \in X_z$ because $(y, \overline{y}) \in X_z$. This implies that $p(y_1, \overline{y}) > 0$, and so, $P(\mathbf{y}, \mathbf{u}) > \hat{k}$ for all $\mathbf{u} \in U^3$. Together, this implies that $P(\mathbf{y}', \mathbf{u}) < P(\mathbf{y}, \mathbf{u})$ for all $\mathbf{u} \in U^3$.

In order to prove the incompatibility, we show that *Pareto* implies that $P(\mathbf{y}', \mathbf{u}') \geq P(\mathbf{y}, \mathbf{u}')$ for the profile $\mathbf{u}' := (u', u', u') \in U^3$. This is indeed an implication of *Pareto* if we have $u'(y_i, \overline{y}) \geq u'(y_i', \overline{y}')$ for all $i \in \{1, 2, 3\}$. We have $u'(y, \overline{y}) \geq u'(z_a, \overline{y}^z)$ because $(y, \overline{y}) \notin X_Q(u')$. This implies that $u'(y_1, \overline{y}) \geq u'(y_1', \overline{y}')$, because $(y_1, \overline{y}) = (y, \overline{y})$ and $(y_1', \overline{y}') = (z_a, \overline{y}^z)$. We have $u'(y_i, \overline{y}) \geq u'(y_i', \overline{y}')$ for all $i \in \{2, 3\}$ because both i's own income and i's relative income is weakly larger under $\mathbf{y}$ than under $\mathbf{y}'$ since $(y_i, \overline{y}) = (\overline{y}, \overline{y})$, $(y_i', \overline{y}') = (\overline{y}^z, \overline{y}^z)$ and $\overline{y} \geq \overline{y}^z$. The desired result.

A3 Proof of Lemma 1

We show that there exists some $j \in N(\mathbf{y})$ for which $y_j < y_j'$ and $\frac{y_j}{\overline{y}} \leq \frac{y_j'}{\overline{y}'}$. If that is the case, then the monotonicity properties of utility functions imply that $u_j(y_j, y_j/\overline{y}) < u_j(y_j', y_j'/\overline{y}')$ because $u_j \in U^B$.

Assume for simplicity that the number of individuals $n(\mathbf{y})$ is odd.[48] As $\overline{y} < \overline{y}'$, more than half of the individuals earn an income no larger than $\overline{y}$ in distribution $\mathbf{y}$, while more than half of the individuals earn an income no smaller than $\overline{y}'$ in distribution $\mathbf{y}'$. Therefore, there must be an individual j with $y_j \leq \overline{y} < \overline{y}' \leq y_j'$. This implies that $y_j < y_j'$ and $\frac{y_j}{\overline{y}} \leq 1 \leq \frac{y_j'}{\overline{y}'}$, as desired.

[47] We define *Weak Pareto* in Section 4.3. P_U satisfies *Weak Pareto* because P_U satisfies *Pareto*. Again, we allow ourselves to simply reference Lemma 4 in order to avoid duplicating the argument.

[48] The argument is the same if $n(\mathbf{y})$ is even, but is less easily exposed.

A4 Proof of Proposition 4

P_U is a fair additive index because P_U satisfies *Domination* (Proposition 1). Furthermore, we have $X_z = X_Q(U)$ because P_U is a fair additive index that satisfies *Weak Pareto* (Lemma 4).

First, assume to the contrary that Condition (i) in the definition of a hierarchical index does not hold; that is, function p is not strictly decreasing in its first argument on X_z. As P_U is a fair additive index, p is weakly decreasing in its first argument on X_z. Thus, the contradiction assumption implies that there exist two bundles $(y, \overline{y}), (y', \overline{y}) \in X_z$ with $0 < y < y' < z(\overline{y})$ such that $p(y, \overline{y}) = p(y', \overline{y})$. As $(y, \overline{y}) \in X_Q(U)$, there exists some $u' \in U$ such that $(y, \overline{y}) \in X_Q(u')$. Let $\lambda' := y'/y$, which is such that $\lambda' > 1$, and let $\overline{y}' := \lambda' \overline{y}$. Bundle $(y', \overline{y}') := (\lambda' y, \lambda' \overline{y})$ is such that $u'(y', \overline{y}') > u'(y, \overline{y})$ because utility is strictly increasing in own income when relative income is constant. As P_U satisfies *Weak Pareto*, Lemma 3 implies that $p(y', \overline{y}') < p(y, \overline{y})$. We also have $p(y', \overline{y}) \leq p(y', \overline{y}')$ because function p is weakly increasing in its second argument and $\overline{y}' \geq \overline{y}$. Transitivity then implies that $p(y', \overline{y}) < p(y, \overline{y})$, which yields the desired contradiction.

There remains to show that Condition (ii) in the definition of a hierarchical index holds; that is, for all $(y, \overline{y}), (y, \overline{y}') \in X_A \cap X_z$, we have $p(y, \overline{y}) = p(y, \overline{y}')$. If $\overline{y} = \overline{y}'$, then that is trivially the case, as P_U is a fair additive index. Thus, consider without loss of generality that $\overline{y} < \overline{y}'$. By assumption, there exists some $u' \in U \cap U^*$. Since $u' \in U^*$, we have $X_A = X_Q(u')$. Therefore, we have $X_A \subseteq X_z$ because $X_z = X_Q(U)$. Consider the contradiction assumption that for some $y \in [0, z_a)$, we have $p(y, \overline{y}) \neq p(y, \overline{y}')$. We must then have $p(y, \overline{y}) < p(y, \overline{y}')$, because function p is weakly increasing in its second argument on X_z, given that P_U is a fair additive index. Now, as $u' \in U^*$, we have $(y, \overline{y}) \in X_Q(u')$ and $u'(y, \overline{y}) = u'(y, \overline{y}')$. We can thus apply Lemma 3, because we also have that P_U satisfies *Weak Pareto* and $\overline{y} < \overline{y}'$. As $u'(y, \overline{y}) = u'(y, \overline{y}')$, Lemma 3 implies that $p(y, \overline{y}) \geq p(y, \overline{y}')$, the desired contradiction.

A5 Proof of Theorem 1

$\Rightarrow$. We show that any $P_{U^{\bar{\sigma}}}$ satisfying these two axioms has the required properties.

As the self-centered preference $u^0 \in U^{\bar{\sigma}}$, we have $U^* \cap U^{\bar{\sigma}} \neq \emptyset$. By Proposition 4, if $P_{U^{\bar{\sigma}}}$ satisfies *Domination* and *Weak Pareto*, then $P_{U^{\bar{\sigma}}}$ is a hierarchical index. By definition of a hierarchical index, we have $X_z = X_Q(U^{\bar{\sigma}})$.

We show that $X_z = X_{z^*}$; that is, $z(\overline{y}) = z^*(\overline{y})$ for all $\overline{y} \geq z_a$. The proof exploits the fact that $X_z = X_Q(U^{\bar{\sigma}})$.

We first show that $z(\overline{y}) = z_a$ for all $\overline{y} \leq \overline{y}^z$. Take any $\overline{y} \leq \overline{y}^z$. We start by showing that $z(\overline{y}) \geq z_a$. For all $y' < z_a$ we have $(y', \overline{y}) \in X_Q(u^0)$, i.e., a self-centered individual is welfare poor when she earns less than the subsistence income. This implies that $(y', \overline{y}) \in X_Q(U^{\bar{\sigma}})$ for all $y' < z_a$ because $u^0 \in U^{\bar{\sigma}}$. Therefore, we have $z(\overline{y}) \geq z_a$ because $X_z = X_Q(U^{\bar{\sigma}})$. There remains to show that $z(\overline{y}) \leq z_a$. We must have $z(\overline{y}) \leq y'$ for any $y' \geq 0$ such that $(y', \overline{y}) \notin X_Q(U^{\bar{\sigma}})$ because $X_z = X_Q(U^{\bar{\sigma}})$. It is thus sufficient to show that $(z_a, \overline{y}) \notin X_Q(U^{\bar{\sigma}})$. We have $u(z_a, \overline{y}) \geq u(z_a, \overline{y}^z)$ for all $u \in U^{\bar{\sigma}}$ because $\overline{y} \leq \overline{y}^z$ and individual utility is weakly decreasing in the median income. By definition of $X_Q(U^{\bar{\sigma}})$, we thus have $(z_a, \overline{y}) \notin X_Q(U^{\bar{\sigma}})$, as desired.

We then show that $z(\overline{y}) = R + R\bar{\sigma}\overline{y}$ for all $\overline{y} > \overline{y}^z$. Take any $\overline{y} > \overline{y}^z$. First, we show that $z(\overline{y}) \leq R + R\bar{\sigma}\overline{y}$. Letting $y := R + R\bar{\sigma}\overline{y}$, we have by Lemma 5, Part (iii), that $(y, \overline{y}) \in X_Q(u^\sigma) \Leftrightarrow \sigma > \sigma^*$ for $\sigma^* := \frac{y - z_a}{z_a\overline{y} - y\overline{y}^z}$. Replacing y and R by their expressions in the definition of σ^* yields $\sigma^* = \bar{\sigma}$. This implies that $(y, \overline{y}) \in X_Q(u^\sigma) \Leftrightarrow \sigma > \bar{\sigma}$. As $\sigma \leq \bar{\sigma}$ for all $u^\sigma \in U^{\bar{\sigma}}$, we have thus shown that $(y, \overline{y}) \notin X_Q(U^{\bar{\sigma}})$. This implies that $z(\overline{y}) \leq y$ because $X_z = X_Q(U^{\bar{\sigma}})$. By the definition of y, we get $z(\overline{y}) \leq R + R\bar{\sigma}\overline{y}$. There remains to show that $z(\overline{y}) \geq R + R\bar{\sigma}\overline{y}$. Letting $y := R + R\bar{\sigma}\overline{y}$, we have by Lemma 5, Part (ii), that $u^{\sigma^*}(y, \overline{y}) = u^{\sigma^*}(z_a, \overline{y}^z)$ for $\sigma^* := \frac{y - z_a}{z_a\overline{y} - y\overline{y}^z}$. This implies that $u^{\bar{\sigma}}(R + R\bar{\sigma}\overline{y}, \overline{y}) = u^{\bar{\sigma}}(z_a, \overline{y}^z)$ because $\sigma^* = \bar{\sigma}$ (as shown above). As utility is strictly increasing in own income, we thus have for any $y' < R + R\bar{\sigma}\overline{y}$ that $u^{\bar{\sigma}}(y', \overline{y}) < u^{\bar{\sigma}}(y, \overline{y})$. By transitivity, this yields $u^{\bar{\sigma}}(y', \overline{y}) < u^{\bar{\sigma}}(z_a, \overline{y}^z)$; that is, $(y', \overline{y}) \in X_Q(u^{\bar{\sigma}})$. Therefore, we have $(y', \overline{y}) \in X_Q(U^{\bar{\sigma}})$ for any $y' < R + R\bar{\sigma}\overline{y}$ because $u^{\bar{\sigma}} \in U^{\bar{\sigma}}$. This shows that $z(\overline{y}) \geq R + R\bar{\sigma}\overline{y}$ because $X_z = X_Q(U^{\bar{\sigma}})$. Hence, we have shown that $z(\overline{y}) = z^*(\overline{y})$ for all $\overline{y} \geq z_a$.

There remains to show that for all $(y, \overline{y}), (y', \overline{y}') \in X_{z^*} \backslash X_A$ with $\frac{y - z_a}{y - \overline{y}^z} = \frac{y' - z_a}{\overline{y}' - \overline{y}^z}$ we have $p(y, \overline{y}) = p(y', \overline{y}')$. Assume to the contrary that there are two bundles $(y, \overline{y}), (y', \overline{y}') \in X_{z^*} \backslash X_A$ with $\frac{y - z_a}{\overline{y} - \overline{y}^z} = \frac{y' - z_a}{\overline{y}' - \overline{y}^z}$ but $p(y, \overline{y}) \neq p(y', \overline{y}')$.

If $\overline{y} = \overline{y}'$, then the two bundles must be identical, a contradiction to $p(y, \overline{y}) \neq p(y', \overline{y}')$, because $P_{U^{\bar{\sigma}}}$ is a fair additive index. Therefore, we have $\overline{y} \neq \overline{y}'$. Without loss of generality, assume that $\overline{y} < \overline{y}'$. We must have $y \geq z_a$ and $y' \geq z_a$ because $(y, \overline{y}), (y', \overline{y}') \notin X_A$. We must also have $\overline{y} > \overline{y}^z$ because $(y, \overline{y}) \in X_{z^*}$ and $y \geq z_a$.

There are two cases:

- Case 1: $p(y, \overline{y}) < p(y', \overline{y}')$.

  This case is such that there exists some $y'' < y$ for which $p(y'', \overline{y}) < p(y', \overline{y}')$. Indeed, function p is continuous on $X_Q(U^{\bar{\sigma}})$ and $(y, \overline{y}) \in X_Q(U^{\bar{\sigma}})$ because $X_Q(U^{\bar{\sigma}}) = X_{z^*}$.

  We show that there exists a preference $u^\sigma \in U^{\bar{\sigma}}$ for which $(y'', \overline{y}) \in X_Q(u^\sigma)$ and $u^\sigma(y'', \overline{y}) < u^\sigma(y', \overline{y}')$. If such a preference exists, we can resort to Lemma 3 because $P_{U^{\bar{\sigma}}}$ is a fair additive index that satisfies *Weak Pareto* and $\overline{y} < \overline{y}'$. Lemma 3 then implies that $p(y'', \overline{y}) > p(y', \overline{y}')$, the desired contradiction.

  There remains to show the existence of such $u^\sigma \in U^{\bar{\sigma}}$. The conditions for Lemma 5 are met by bundle $(y', \overline{y}')$. Indeed, we have $\overline{y}' > \overline{y}^z$ because $\overline{y}' > \overline{y} > \overline{y}^z$. We also have $y' \in [z_a, R + R\bar{\sigma}\overline{y}']$ because $(y', \overline{y}') \in X_{z^*} \backslash X_A$. By Lemma 5, Part (ii), we have $u^{\sigma^*}(y', \overline{y}') = u^{\sigma^*}(z_a, \overline{y}^z)$ for $\sigma^* := \frac{y' - z_a}{z_a \overline{y}' - y' \overline{y}^z}$. As $\frac{y - z_a}{\overline{y} - \overline{y}^z} = \frac{y' - z_a}{\overline{y}' - \overline{y}^z}$, Lemma 5, Part (iv), further implies that $u^{\sigma^*}(y, \overline{y}) = u^{\sigma^*}(y', \overline{y}')$ for the same σ^*.[49] Therefore, we have $u^{\sigma^*}(y'', \overline{y}) < u^{\sigma^*}(z_a, \overline{y}^z)$ because $y'' < y$. This shows that $(y'', \overline{y}) \in X_Q(u^{\sigma^*})$. By transitivity, we also have $u^{\sigma^*}(y'', \overline{y}) < u^{\sigma^*}(y', \overline{y}')$. Finally, we have $u^{\sigma^*} \in U^{\bar{\sigma}}$ because Lemma 5, Part (i), implies that $\sigma^* \in [0, \bar{\sigma}]$, which proves that u^{σ^*} has the required properties.

- Case 2: $p(y, \overline{y}) > p(y', \overline{y}')$.

  This case is such that there exists some $y'' > y$ for which $p(y'', \overline{y}) > p(y', \overline{y}')$. Indeed, function p is continuous on (the open) set $X_Q(U^{\bar{\sigma}})$, and $(y, \overline{y}) \in X_Q(U^{\bar{\sigma}})$ because $X_Q(U^{\bar{\sigma}}) = X_{z^*}$. We must have $p(y'', \overline{y}) > 0$ because $p(y', \overline{y}') \geq 0$ as $P_{U^{\bar{\sigma}}}$ is a fair additive index. This, in turn, implies that $y'' < z^*(\overline{y})$ and thus $y'' \in [z_a, R + R\bar{\sigma}\overline{y})$.

  Consider the two distributions $\mathbf{y} := (y'', \overline{y}, \overline{y})$ and $\mathbf{y}' := (y', \overline{y}', \overline{y}')$, which are, respectively, such that $\overline{\mathbf{y}} = \overline{y}$ and $\overline{\mathbf{y}}' = \overline{y}'$. By construction, we have $(y_1, \overline{y}) = (y'', \overline{y})$, $(y_1', \overline{y}') = (y', \overline{y}')$. Also, we have $(y_i, \overline{y}) = (\overline{y}, \overline{y})$ and $(y_i', \overline{y}') = (\overline{y}', \overline{y}')$ for all $i \in \{2, 3\}$. We have that bundles $(\overline{y}, \overline{y}), (\overline{y}', \overline{y}') \notin X_Q(U^{\bar{\sigma}})$ (Lemma 2). Therefore, $p(\overline{y}, \overline{y}) = p(\overline{y}', \overline{y}') = 0$ because $P_{U^{\bar{\sigma}}}$ is a fair additive index. Consider any $\mathbf{u} := (u, u, u)$. If we have $P_{U^{\bar{\sigma}}}(\mathbf{y}, \mathbf{u}) \leq P_{U^{\bar{\sigma}}}(\mathbf{y}', \mathbf{u})$, then we get a contradiction to $p(y'', \overline{y}) > p(y', \overline{y}')$. Indeed, $P_{U^{\bar{\sigma}}}(\mathbf{y}, \mathbf{u}) \leq P_{U^{\bar{\sigma}}}(\mathbf{y}', \mathbf{u})$ implies that $p(y'', \overline{y}) \leq p(y', \overline{y}')$ because $p(y_i, \overline{y}) = p(y_i', \overline{y}') = 0$ for all $i \in \{2, 3\}$.

[49] The conditions for Lemma 5, Part (iv), are met because $y \geq z_a$ and $\overline{y} > \overline{y}^z$.

We show that *Domination* implies that $P_{U^{\bar{\sigma}}}(\mathbf{y},\mathbf{u}) \leq P_{U^{\bar{\sigma}}}(\mathbf{y}',\mathbf{u})$. *Domination* implies $P_{U^{\bar{\sigma}}}(\mathbf{y},\mathbf{u}) \leq P_{U^{\bar{\sigma}}}(\mathbf{y}',\mathbf{u})$ if $u^{\sigma}(y_i,\overline{y}) \geq u^{\sigma}(y_i',\overline{y}')$ for all $i \in \{1,2,3\}$ and all $u^{\sigma} \in U^{\bar{\sigma}}$ such that $(y_i,\overline{y}) \in X_Q(u^{\sigma})$. As $(\overline{y},\overline{y}),(\overline{y}',\overline{y}') \notin X_Q(U^{\bar{\sigma}})$, there remains to show that $u^{\sigma}(y'',\overline{y}) \geq u^{\sigma}(y',\overline{y}')$ for all $u^{\sigma} \in U^{\bar{\sigma}}$ for which $(y'',\overline{y}) \in X_Q(u^{\sigma})$. Lemma 5, Part (iii), implies that $(y'',\overline{y}) \in X_Q(u^{\sigma})$ if and only if $\sigma > \sigma^*$, where $\sigma^* := \frac{y''-z_a}{z_a\overline{y}-y''\overline{y}^z}$.[50] Hence, we must show that $u^{\sigma}(y'',\overline{y}) \geq u^{\sigma}(y',\overline{y}')$ for all $u^{\sigma} \in U^{\bar{\sigma}}$ for which $\sigma > \sigma^*$.

Let the income level $y^e \geq 0$ be defined as $\frac{y^e-z_a}{\overline{y}'-\overline{y}^z} = \frac{y''-z_a}{\overline{y}-\overline{y}^z}$. This definition is such that $y^e > y'$ because $y'' > y$ and $\frac{y'-z_a}{\overline{y}'-\overline{y}^z} = \frac{y-z_a}{\overline{y}-\overline{y}^z}$. As $\frac{y^e-z_a}{\overline{y}'-\overline{y}^z} = \frac{y''-z_a}{\overline{y}-\overline{y}^z}$, Lemma 5, Part (iv), implies that $u^{\sigma^*}(y^e,\overline{y}') = u^{\sigma^*}(y'',\overline{y})$ for the same σ^*. As $\overline{y} < \overline{y}'$, Lemma 5, Part (v), further implies that $u^{\sigma}(y^e,\overline{y}') < u^{\sigma}(y'',\overline{y})$ for all $\sigma > \sigma^*$. We also have $u^{\sigma}(y',\overline{y}') < u^{\sigma}(y^e,\overline{y}')$ for all $\sigma \geq 0$ because $y^e > y'$. By transitivity, we get $u^{\sigma}(y',\overline{y}') < u^{\sigma}(y'',\overline{y})$ for all $\sigma > \sigma^*$, the desired result.

$\Leftarrow$. We show that $P_{U^{\bar{\sigma}}}$ satisfies the two axioms.

Domination: Take any $(\mathbf{y},\mathbf{u}),(\mathbf{y}',\mathbf{u}') \in \mathcal{X}_{U^{\bar{\sigma}}}$ that satisfy the preconditions under which *Domination* implies that $P_{U^{\bar{\sigma}}}(\mathbf{y}',\mathbf{u}') \leq P_{U^{\bar{\sigma}}}(\mathbf{y},\mathbf{u})$; that is, we have $n(\mathbf{y}) = n(\mathbf{y}')$ and $u(y_i',\overline{y}') \geq u(y_i,\overline{y})$ for all $i \in N(\mathbf{y}')$ and all $u \in U^{\bar{\sigma}}$ such that $(y_i',\overline{y}') \in X_Q(u)$. In order to prove that $P_{U^{\bar{\sigma}}}(\mathbf{y}',\mathbf{u}') \leq P_{U^{\bar{\sigma}}}(\mathbf{y},\mathbf{u})$, we show that $p(y_i',\overline{y}') \leq p(y_i,\overline{y})$ for all $i \in N(\mathbf{y}')$.

Take any $i \in N(\mathbf{y}')$ for whom $(y_i',\overline{y}') \notin X_{z^*}$. Since $P_{U^{\bar{\sigma}}}$ is a fair additive index, we have $p(y_i',\overline{y}') = 0$ and thus $p(y_i',\overline{y}') \leq p(y_i,\overline{y})$.

Take any $i \in N(\mathbf{y}')$ for whom $(y_i',\overline{y}') \in X_A$. If $z_a > y_i' \geq y_i$, then we directly have $p(y_i',\overline{y}') \leq p(y_i,\overline{y})$ because $P_{U^{\bar{\sigma}}}$ is a hierarchical index. We have $z_a > y_i'$, as $(y_i',\overline{y}') \in X_A$. There remains to show that $y_i' \geq y_i$. As $z_a > y_i'$, we have $(y_i',\overline{y}') \in X_Q(u^0)$ for the self-centered preference u^0. The precondition then implies that $u^0(y_i',\overline{y}') \geq u^0(y_i,\overline{y})$ because $u^0 \in U^{\bar{\sigma}}$. This inequality implies that $y_i' \geq y_i$, as desired.

Finally, take any $i \in N(\mathbf{y}')$ for whom $(y_i',\overline{y}') \in X_{z^*}\backslash X_A$. If $y_i < z_a$, then we directly have that $p(y_i',\overline{y}') \leq p(y_i,\overline{y})$ because $y_i' \geq z_a$, as $(y_i',\overline{y}') \notin X_A$ and $P_{U^{\bar{\sigma}}}$ is a hierarchical index. There remains the alternative case $y_i \geq z_a$. If we have $\frac{y_i'-z_a}{\overline{y}'-\overline{y}^z} \geq \frac{y_i-z_a}{\overline{y}-\overline{y}^z}$, then the properties of function p on $X_{z^*}\backslash X_A$ imply that $p(y_i',\overline{y}') \leq p(y_i,\overline{y})$, as desired. Assume to the contrary that $\frac{y_i'-z_a}{\overline{y}'-\overline{y}^z} < \frac{y_i-z_a}{\overline{y}-\overline{y}^z}$. We show that there exists some $u^{\sigma} \in U^{\bar{\sigma}}$ for which $(y_i',\overline{y}') \in X_Q(u^{\sigma})$ and $u^{\sigma}(y_i,\overline{y}) > u^{\sigma}(y_i',\overline{y}')$, a contradiction to the precondition of *Domination*. There are two cases.

[50] The conditions for Lemma 5 are met because $y'' \in [z_a, R + R\bar{\sigma}\overline{y})$ and $\overline{y} > \overline{y}^z$.

- Case 1: $z_a \leq y_i < z^*(\overline{y})$.

  This case is such that $\overline{y} > \overline{y}^z$, because $z_a < z^*(\overline{y})$, and we also have $\overline{y}' > \overline{y}^z$ because $(y_i', \overline{y}') \in X_{z^*} \backslash X_A$. By Lemma 5, Part (ii), we have $u^{\sigma^*}(y_i, \overline{y}) = u^{\sigma^*}(z_a, \overline{y}^z)$ for $\sigma^* := \frac{y_i - z_a}{z_a \overline{y} - y_i \overline{y}^z}$. Let the income level $y^e \geq 0$ be defined as $\frac{y^e - z_a}{\overline{y}' - \overline{y}^z} = \frac{y_i - z_a}{\overline{y} - \overline{y}^z}$. This definition is such that $y^e > y_i'$ because $\frac{y_i' - z_a}{\overline{y}' - \overline{y}^z} < \frac{y_i - z_a}{\overline{y} - \overline{y}^z}$. As $\frac{y^e - z_a}{\overline{y}' - \overline{y}^z} = \frac{y_i - z_a}{\overline{y} - \overline{y}^z}$, Lemma 5, Part (iv), further implies that $u^{\sigma^*}(y^e, \overline{y}') = u^{\sigma^*}(y_i, \overline{y})$ for the same σ^*. As $y^e > y_i'$, this yields $u^{\sigma^*}(y_i', \overline{y}') < u^{\sigma^*}(y_i, \overline{y})$. By transitivity, we also get $u^{\sigma^*}(y_i', \overline{y}') < u^{\sigma^*}(z_a, \overline{y}^z)$ and thus $(y_i', \overline{y}') \in X_Q(u^{\sigma^*})$. Preference u^{σ^*} thus has the required properties because $u^{\sigma^*} \in U^{\bar{\sigma}}$, as $\sigma^* \in [0, \bar{\sigma}]$ (Lemma 5, Part (i)).

- Case 2: $z_a \leq z^*(\overline{y}) \leq y_i$.

  As $(y_i', \overline{y}') \in X_{z^*}$, there exists some preference $u^\sigma \in U^{\bar{\sigma}}$ for which $(y_i', \overline{y}') \in X_Q(u^\sigma)$ because $X_{z^*} = X_Q(U^{\bar{\sigma}})$. By definition of $X_Q(u^\sigma)$, we have $u^\sigma(z_a, \overline{y}^z) > u^\sigma(y_i', \overline{y}')$. We must have $u^\sigma(z_a, \overline{y}^z) \leq u^\sigma(y_i, \overline{y})$ because this case is such that $(y_i, \overline{y}) \notin X_{z^*}$ and $X_{z^*} = X_Q(U^{\bar{\sigma}})$. By transitivity, we have $u^\sigma(y_i, \overline{y}) > u^\sigma(y_i', \overline{y}')$. Preference $u^\sigma \in U^{\bar{\sigma}}$ thus has all the desired properties.

Weak Pareto: Take any $(\mathbf{y}, \mathbf{u}), (\mathbf{y}', \mathbf{u}') \in \mathcal{X}_{U^{\bar{\sigma}}}$ that satisfy the preconditions under which *Weak Pareto* implies $P_{U^{\bar{\sigma}}}(\mathbf{y}', \mathbf{u}') \leq P_{U^{\bar{\sigma}}}(\mathbf{y}, \mathbf{u})$. That is, we have $n(\mathbf{y}) = n(\mathbf{y}')$, $u_i(y_i', \overline{y}') \geq u_i(y_i, \overline{y})$ for all $i \in N(\mathbf{y}')$, and $y_j' \geq \frac{\overline{y}'}{\overline{y}} y_j$ for all $j \notin Q(\mathbf{y}, \mathbf{u})$. The unanimous preference for distribution $\mathbf{y}'$ implies that $\overline{y} \leq \overline{y}'$ (Lemma 1). In order to prove $P_{U^{\bar{\sigma}}}(\mathbf{y}', \mathbf{u}') \leq P_{U^{\bar{\sigma}}}(\mathbf{y}, \mathbf{u})$, we show that $p(y_i', \overline{y}') \leq p(y_i, \overline{y})$ for all $i \in N(\mathbf{y}')$.

Take any $i \in N(\mathbf{y}')$ for whom $(y_i', \overline{y}') \notin X_{z^*}$. Since $P_{U^{\bar{\sigma}}}$ is a fair additive index, we have $p(y_i', \overline{y}') = 0$ and thus $p(y_i', \overline{y}') \leq p(y_i, \overline{y})$.

Take any $i \in N(\mathbf{y}')$ for whom $(y_i', \overline{y}') \in X_A$. We first show that $y_i \leq y_i'$. This follows from $u_i(y_i', \overline{y}') \geq u_i(y_i, \overline{y})$, because $\overline{y} \leq \overline{y}'$ and utility functions are non-increasing in the median income. This implies that $y_i \leq y_i' < z_a$ because $(y_i', \overline{y}') \in X_A$. We therefore have $p(y_i', \overline{y}') \leq p(y_i, \overline{y})$, because $P_{U^{\bar{\sigma}}}$ is hierarchical.

Finally, take any $i \in N(\mathbf{y}')$ for whom $(y_i', \overline{y}') \in X_{z^*} \backslash X_A$. We have $y_i' \geq z_a$ because $(y_i', \overline{y}') \notin X_A$. We also have $\overline{y}' > \overline{y}^z$, because $(y_i', \overline{y}') \in X_{z^*} \backslash X_A$. If $y_i < z_a$, then we directly have $p(y_i', \overline{y}') \leq p(y_i, \overline{y})$, because $P_{U^{\bar{\sigma}}}$ is a hierarchical index. There remains the case $y_i \geq z_a$. If we have *both* $(y_i, \overline{y}) \in X_{z^*} \backslash X_A$ and

$$\frac{y_i' - z_a}{\overline{y}' - \overline{y}^z} \geq \frac{y_i - z_a}{\overline{y} - \overline{y}^z}, \tag{E.3}$$

then the properties of function p imply that $p(y_i', \overline{y}') \leq p(y_i, \overline{y})$, the desired result. There remains to show that *both* hold.

We first show that there exists some $u^{\hat{\sigma}} \in U^{\bar{\sigma}}$ for which $(y_i, \overline{y}) \in X_Q(u^{\hat{\sigma}})$ and $u^{\hat{\sigma}}(y_i', \overline{y}') \geq u^{\hat{\sigma}}(y_i, \overline{y})$. First, assume that $i \notin Q(\mathbf{y}, \mathbf{u})$. There exists some $u^{\hat{\sigma}} \in U^{\bar{\sigma}}$ for which $(y_i', \overline{y}') \in X_Q(u^{\hat{\sigma}})$ because $(y_i', \overline{y}') \in X_{z^*} = X_Q(U^{\bar{\sigma}})$. The precondition of *Weak Pareto* requires that $y_i' \geq \frac{\overline{y}'}{\overline{y}} y_i$ because $i \notin Q(\mathbf{y}, \mathbf{u})$. We thus have $u^{\hat{\sigma}}(y_i', \overline{y}') \geq u^{\hat{\sigma}}(y_i, \overline{y})$, because $y_i' \geq \frac{\overline{y}'}{\overline{y}} y_i$, $\frac{\overline{y}'}{\overline{y}} \geq 1$ and utility is increasing in own income when holding relative income constant. We also have $(y_i, \overline{y}) \in X_Q(u^{\hat{\sigma}})$ because $(y_i', \overline{y}') \in X_Q(u^{\hat{\sigma}})$ and $u^{\hat{\sigma}}(y_i', \overline{y}') \geq u^{\hat{\sigma}}(y_i, \overline{y})$, showing that $u^{\hat{\sigma}}$ has all the desired properties. Second, assume that $i \in Q(\mathbf{y}, \mathbf{u})$. Consider $u^{\hat{\sigma}} := u_i$. We have $(y_i, \overline{y}) \in X_Q(u^{\hat{\sigma}})$ because $i \in Q(\mathbf{y}, \mathbf{u})$. We further have $u^{\hat{\sigma}}(y_i', \overline{y}') \geq u^{\hat{\sigma}}(y_i, \overline{y})$, because the preconditions of *Weak Pareto* imply that $u_i(y_i', \overline{y}') \geq u_i(y_i, \overline{y})$, i.e., $u^{\hat{\sigma}}$ has all the desired properties.

We have $(y_i, \overline{y}) \in X_{z^*}$ because $X_{z^*} = X_Q(U^{\bar{\sigma}})$, and there exists some $u^{\hat{\sigma}} \in U^{\bar{\sigma}}$ for which $(y_i, \overline{y}) \in X_Q(u^{\hat{\sigma}})$. This yields $(y_i, \overline{y}) \in X_{z^*} \backslash X_A$ because $y_i \geq z_a$. There remains to show that Inequality (E.3) holds. If $\overline{y} = \overline{y}'$, then Inequality (E.3) directly follows because $u^{\hat{\sigma}}(y_i', \overline{y}') \geq u^{\hat{\sigma}}(y_i, \overline{y})$. If, instead, $\overline{y} \neq \overline{y}'$, then we have $\overline{y} < \overline{y}'$ because $\overline{y} \leq \overline{y}'$. We have shown above that $(y_i, \overline{y}) \in X_Q(u^{\hat{\sigma}})$ and $u^{\hat{\sigma}}(y_i', \overline{y}') \geq u^{\hat{\sigma}}(y_i, \overline{y})$. By Lemma 5, Part (iii), we have $(y_i, \overline{y}) \in X_Q(u^{\hat{\sigma}}) \Leftrightarrow \hat{\sigma} > \sigma^*$ for $\sigma^* := \frac{y_i - z_a}{z_a \overline{y} - y_i \overline{y}^z}$.[51] Thus, we have $\hat{\sigma} > \sigma^*$. Let the income level $y^e \geq 0$ be defined as $\frac{y^e - z_a}{\overline{y}' - \overline{y}^z} = \frac{y_i - z_a}{\overline{y} - \overline{y}^z}$. Lemma 5, Part (iv), implies $u^{\sigma^*}(y^e, \overline{y}') = u^{\sigma^*}(y_i, \overline{y})$ for the same σ^*. As $\overline{y} < \overline{y}'$, Lemma 5, Part (v), further implies that $u^{\hat{\sigma}}(y^e, \overline{y}') < u^{\hat{\sigma}}(y_i, \overline{y})$, because $\hat{\sigma} > \sigma^*$. We must thus have $y_i' \geq y^e$, because $u^{\hat{\sigma}}(y_i', \overline{y}') \geq u^{\hat{\sigma}}(y_i, \overline{y})$. Inequality (E.3) then directly follows from the fact that $y_i' \geq y^e$ and $\frac{y^e - z_a}{\overline{y}' - \overline{y}^z} = \frac{y_i - z_a}{\overline{y} - \overline{y}^z}$, the desired result.

Take any $(\mathbf{y}, \mathbf{u}), (\mathbf{y}', \mathbf{u}') \in \mathcal{X}_{U^{\bar{\sigma}}}$ that satisfy the preconditions under which *Weak Pareto* implies $P_{U^{\bar{\sigma}}}(\mathbf{y}', \mathbf{u}') < P_{U^{\bar{\sigma}}}(\mathbf{y}, \mathbf{u})$. The proof is a straightforward adaptation of arguments used above and is thus omitted.

[51] Lemma 5 applies because $y_i \in [z_a, z^*(\overline{y})]$ and $\overline{y} > \overline{y}^z$ because $(y_i, \overline{y}) \in X_{z^*} \backslash X_A$.

Online Appendix
Global Income Poverty Measurement with Preference Heterogeneity: Theory and Application

Benoit Decerf Mery Ferrando Natalie Naïri Quinn

January 11, 2023

S1 Relationship with the Capability Approach

In this section, we explain in more details that the individual's preference over own income and relative income, which are the basis of our framework, can be understood as a "reduced form" for the individual's preference over underlying functionings (unrelated to misanthropic feelings). We also show that our definition of the welfare poor, based on a reference bundle, is equivalent to a definition of the welfare poor based on a reference capability (Sen, 1985). Our framework builds on the theoretical foundations provided by Ravallion (2020) for global income poverty measurement.[1] As we build on Ravallion (2020), our premises are slightly different from those of the approach of Atkinson and Bourguignon (2001).[2]

Sen argues that the space in which individual welfare should be measured is the space of functionings and capabilities. Functionings are, loosely speaking, what a person can do and be. In turn, capabilities are sets of functioning vectors, i.e. opportunity sets in the space of functionings.

After Atkinson and Bourguignon (2001), two major functionings are deemed key for the measurement of global poverty. First is *subsistence*, which is typically implemented through nutritional status. In a first approximation, the real cost of consuming a given amount of calories does not evolve with a society's median

[1] Unlike Ravallion (2020), who surveys some of the major issues in global poverty measurement, we ignore here important problems associated to prices and individual characteristics other than preferences.

[2] As explained by Ravallion (2020), Atkinson and Bourguignon (2001) take a non-welfarist approach that does not consider the trade-offs that individuals make between different functionings. Instead, their approach respect the functioning that is deemed relevant in a given society, without demanding welfare-consistency.

income. Hence, the larger an individual's income, the higher the number of calories she can purchase, independently of her society's median income. Second is *social inclusion*, which can be achieved through the consumption of clothing and housing, as well as food diets.[3] The level of social participation that a given bundle of goods provides depends on the society's median income. Typically, the larger the median income, the smaller the level of social participation that an individual can reach with fixed income. This idea borrows from Smith (1776) and Townsend (1979): individuals whose income is too far below median income in their society are at risk of social exclusion because their income might not be sufficient for them to participate in the everyday activities of their society.

Let $f = (f_1, f_2)$ denote a vector of functionings, where the first component f_1 captures the level of nutrition and the second component f_2 captures the level of social participation. To be sure, each of these components reflects a continuous score on its associated functioning, and not a binary status. Individual i's budget set in the space of functionings $B(y_i, \overline{y})$, i.e. her capability, depends on her income y_i and the median income $\overline{y}$ in her society. This capability is the set of functioning vectors she can potentially achieve. Given this capability $B(y_i, \overline{y})$, the vector of functionings she achieves $f_i^*(y_i, \overline{y})$ depends on her expenditure choices. Individual i could achieve a relatively high level of nutrition if she spends large amounts on cheap calories. Alternatively, she could reach a relatively high level of social participation is she spends large amounts on clothes and housing, or expensive calories.

The expenditure choices of i depends on her primal utility function $w_i(f_i)$, which captures the trade-off she makes between different functionings. We extend the theoretical framework laid out in Ravallion (2020) by allowing this primal utility function to be individual specific, whereas he assumes this function to be stable and interpersonally comparable. We argue that this extension is called for when individuals with the same capability select different expenditure patterns, thereby revealing the different trade-offs they make between nutrition and social participation.

The primal utility function implies that

$$f_i^*(y_i, \overline{y}) = \arg \max_{f_i \in B(y_i, \overline{y})} w_i(f_i).$$

Our analysis based on utility functions u_i is bridged to the primal utility function

[3] As discussed by Ravallion (2020), food consumption can also play a role in social inclusion. Poor individuals do not systematically consume the cheapest calories.

w_i when

$$u_i(y_i, \overline{y}) = w_i(f_i^*(y_i, \overline{y})).$$

In our extended framework, we do *not* assume comparability of the primal utility function across individuals. Rather, interpersonal comparisons emerge from the reference bundle. Our approach based on a reference bundle $(z_a, \overline{y}^z)$ is equivalent to the selection of a reference capability $B(z_a, \overline{y}^z)$. Individual i is welfare poor if her capability $B(y_i, \overline{y})$ does not allow reaching the level of primal welfare that she could reach under the reference capability $B(z_a, \overline{y}^z)$. Hence, individual i is welfare poor if her capability $B(y_i, \overline{y})$ does not contain a vector of functionings that provides her a level of welfare as large as $f_i^*(z_a, \overline{y}^z)$, i.e. if

$$w_i(f_i^*(y_i, \overline{y})) < w_i(f_i^*(z_a, \overline{y}^z)).$$

Observe that two individuals i and j who have different primal utility functions w_i and w_j may have different reference vectors of functionings, i.e. $f_i^*(z_a, \overline{y}^z) \neq f_j^*(z_a, \overline{y}^z)$. However, when $w_i = w_j$, as assumed by Ravallion (2020), we have $f_i^*(z_a, \overline{y}^z) = f_j^*(z_a, \overline{y}^z)$. In that case, assuming a reference bundle $(z_a, \overline{y}^z)$, i.e. assuming a reference capability, is the same as assuming a reference vector of functioning f^*, as he proceeds.

S2 Relation Between Index HH_S and $\mathbb{EH}$

Assume that individual utility functions are distributed i.i.d in the set $U^{\bar{\sigma}}$, which is the subset of utility functions u^{σ} for which $0 \leq \sigma < \bar{\sigma}$.[4] Each individual i draws u_i in $U^{\bar{\sigma}}$ according to some probability measure $\mathbb{P} : \mathcal{F} \to [0, 1]$, where $\mathcal{F}$ is a sigma-algebra on $U^{\bar{\sigma}}$.[5]

Proposition S.1.
Consider any $\bar{\sigma} \in [0, \infty)$. There exist a sigma-algebra $\mathcal{F}$ on $U^{\bar{\sigma}}$ and a probability measure $\mathbb{P}^* : \mathcal{F} \to [0, 1]$ such that

$$HH_S(\mathbf{y}) = \mathbb{E}^* \left(\frac{\#Q(\mathbf{y}, \mathbf{u})}{n(\mathbf{y})} \right) \qquad \textit{for all } \mathbf{y} \in Y^n \textit{ with } \overline{y} \geq \overline{y}^z,$$

[4] That is, individual utility functions are drawn independently from the bundles consumed and independently from all other individuals' utility functions.

[5] $\mathcal{F}$ is thus a set of subsets of U that contains U and is closed under complements and countable unions; the probability measure satisfies $\mathbb{P}(U) = 1$, $\mathbb{P}(\emptyset) = 0$ and is additive over countable collections of pairwise disjoint elements of $\mathcal{F}$.

where $\mathbb{E}^*$ is the expectation operator under $\mathbb{P}^*$ and $\#$ is the set cardinality operator.

Proof. The proof is by construction.

First, we define the probability space $(U^{\bar{\sigma}}, \mathcal{F}, \mathbb{P}^*)$. Let $U^x := \{u^\sigma \in U^{\bar{\sigma}} | 0 \leq \sigma \leq x\}$, let $F := \{U^x | x \in [0, \bar{\sigma}]\}$, and let $\mathcal{F}$ be the closure of F under complements and countable unions. Define $\mathbb{P}^* : F \to [0,1]$ such that

$$\mathbb{P}^*(U^x) := \frac{x}{1 + x\bar{y}^z} \frac{1 + \bar{\sigma}\bar{y}^z}{\bar{\sigma}},$$

interpreted as the probability that $\sigma \leq x$, and extend $\mathbb{P}^*$ to all events in $\mathcal{F}$ through its additivity property.[6]

Second, we show that this probability space has the required property. Recall we can write $u^\sigma(y, \bar{y}) = -\frac{1+\sigma\bar{y}}{y}$. Utility function u^σ is strictly increasing in $y \geq 0$ and decreasing in $\bar{y} \geq 0$ and $\sigma \geq 0$. Let the income level $\zeta(\bar{y}) := \frac{z_a(1+\bar{\sigma}\bar{y})}{1+\bar{\sigma}\bar{y}^z}$, noting that $u^{\bar{\sigma}}(\zeta(\bar{y}), \bar{y}) = u^{\bar{\sigma}}(z_a, \bar{y}^z)$ for all $\bar{y} \geq 0$ and, furthermore, that $\zeta(\bar{y})$ is the highest income level for which an indifference curve of some $u^\sigma \in U^{\bar{\sigma}}$ passes through both bundles $(\zeta(\bar{y}), \bar{y})$ and $(z_a, \bar{y}^z)$. Given any $\mathbf{y} \in Y^n$ with $\bar{y} \geq \bar{y}^z$, note that $\zeta(\bar{y}) = z(\bar{y})$, where $z(\bar{y})$ is the function such that $X_z = X_Q(U^{\bar{\sigma}})$. Letting $\mathbb{I}$ be the indicator function, taking the value 1 if its argument is true and 0 if false, we have

$$\begin{aligned}
\mathbb{E}^* \left(\frac{\#Q(\mathbf{y}, \mathbf{u})}{n(\mathbf{y})} \right) &= \frac{1}{n(\mathbf{y})} \mathbb{E}^* \left(\sum_{i=1}^{n(\mathbf{y})} \mathbb{I}\left(u_i(y_i, \bar{y}) < u_i(z_a, \bar{y}^z)\right) \right) \\
&= \frac{1}{n(\mathbf{y})} \sum_{i=1}^{n(\mathbf{y})} \mathbb{P}^* \left(u_i(y_i, \bar{y}) < u_i(z_a, \bar{y}^z)\right)
\end{aligned}$$

because $n(\mathbf{y})$ is non-random and by linearity of expectation. There are three cases to consider.

Case 1. If $y_i < z_a$ then $u^\sigma(y_i, \bar{y}) < u^\sigma(z_a, \bar{y}) \leq u^\sigma(z_a, \bar{y}^z)$ for all $\sigma \geq 0$ because $\bar{y} \geq \bar{y}^z$, so $\{u^\sigma \in U^{\bar{\sigma}} | u^\sigma(y_i, \bar{y}) < u^\sigma(z_a, \bar{y}^z)\} = U^{\bar{\sigma}}$ and $\mathbb{P}^*\left(u_i(y_i, \bar{y}) < u_i(z_a, \bar{y}^z)\right) = \mathbb{P}^*(U^{\bar{\sigma}}) = 1 = p^{HH_S}(y_i, \bar{y})$.

Case 2. If $y_i \geq z(\bar{y})$ then $u^\sigma(y_i, \bar{y}) \geq u^\sigma(z(\bar{y}), \bar{y}) \geq u^{\bar{\sigma}}(z(\bar{y}), \bar{y}) = u^{\bar{\sigma}}(z_a, \bar{y}^z)$ for all $\sigma \in [0, \bar{\sigma}]$, so $\{u^\sigma \in U^{\bar{\sigma}} | u^\sigma(y_i, \bar{y}) < u^\sigma(z_a, \bar{y}^z)\} = \emptyset$ and $\mathbb{P}^*\left(u_i(y_i, \bar{y}) < u_i(z_a, \bar{y}^z)\right) = \mathbb{P}^*(\emptyset) = 0 = p^{HH_S}(y_i, \bar{y})$.

[6] Note that $S : U^{\bar{\sigma}} \to [0, \bar{\sigma}]$ such that $S : u^\sigma \mapsto \sigma$ is a random variable on this probability space, whose probability density function is smoothly decreasing in σ.

Case 3. If $z_a \leq y_i < z(\bar{y})$ then $u^\sigma(y_i, \bar{y}) < u^\sigma(z_a, \bar{y}^z)$ if and only if $\sigma(z_a\bar{y} - y_i\bar{y}^z) > y_i - z_a$ if and only if $\sigma > \frac{y_i - z_a}{z_a\bar{y} - y_i\bar{y}^z}$, as $\frac{y_i}{\bar{y}} < \frac{z(\bar{y})}{\bar{y}} \leq \frac{z_a}{\bar{y}^z}$. So $\{u^\sigma \in U^{\bar{\sigma}} | u^\sigma(y_i, \bar{y}) < u^\sigma(z_a, \bar{y}^z)\} = U^{\bar{\sigma}} \setminus U^{x_i}$ where $x_i = \frac{y_i - z_a}{z_a\bar{y} - y_i\bar{y}^z}$, and $\mathbb{P}^*(u_i(y_i, \bar{y}) < u_i(z_a, \bar{y}^z)) = 1 - \mathbb{P}^*(U^{x_i})$ which, after straightforward manipulations, can be shown to equal $\frac{z(\bar{y}) - y_i}{z(\bar{y}) - z_a} = p^{HH_S}(y_i, \bar{y})$.

In all three cases we have established that $\mathbb{P}^*(u_i(y_i, \bar{y}) < u_i(z_a, \bar{y}^z)) = p^{HH_S}(y_i, \bar{y})$, so

$$\mathbb{E}^* \left(\frac{\#Q(\mathbf{y}, \mathbf{u})}{n(\mathbf{y})} \right) = \frac{1}{n(\mathbf{y})} \sum_{i=1}^{n(\mathbf{y})} p^{HH_S}(y_i, \bar{y}) = HH_S(\mathbf{y}).$$

■

Proposition S.1 relates to a strain of research on the uncertain identification of the poor. The literature on fuzzy poverty measures starts from the assumption that the poverty line lies in some income range, but its exact value is not precisely known (Cerioli and Zani, 1990). One possible reason is that people have different perceptions about what constitutes poverty (Zheng, 2015). Proposition S.1 has strong similarities with that approach if we assume that the individual utility functions are not precisely known. Indeed, all individuals below the global line are attributed positive poverty scores, even if they might not be welfare poor. Our results are conceptually different because we investigate the trade-offs between own income and relative income. Also, the trade-offs that we characterize do not depend on a probability distribution on individual preferences.

S3 Index HH_S satisfies a minimal version of *Pareto*

In this appendix, we refer to material presented in sections 3, 4 and 5.

No fair additive index is fully welfare-consistent when preferences are heterogeneous (Proposition 3). We show that, in contrast to H_S, HH_S is minimally welfare-consistent and thus grants a minimal role to preferences.

Formally, the (societal) headcount ratio violates *Minimal Pareto*. This minimal welfare-consistency property is a weakening of *Weak Pareto*.[7] For poverty to be reduced, *Minimal Pareto* adds the precondition that a welfare poor individual ℓ escapes welfare poverty and earns an income above the subsistence income z_a.

Axiom S.1 (*Minimal Pareto*).
For all $(\mathbf{y},\mathbf{u}),(\mathbf{y}',\mathbf{u}) \in \mathcal{X}_U$ such that $n(\mathbf{y}) = n(\mathbf{y}')$, if $u_i(y_i', \overline{y}') \geq u_i(y_i, \overline{y})$ for all $i \in N(\mathbf{y})$, $y_j' \geq \frac{\overline{y}'}{\overline{y}} y_j$ for all $j \notin Q(\mathbf{y},\mathbf{u})$ and there is some $\ell \in Q(\mathbf{y},\mathbf{u})$ for whom $\ell \notin Q(\mathbf{y}',\mathbf{u})$ and $y_\ell' > z_a$, then $P_U(\mathbf{y}',\mathbf{u}) < P_U(\mathbf{y},\mathbf{u})$.

Since *Minimal Pareto* is a weakening of *Weak Pareto*, all indices characterized in Theorems 1, 2 and S.1 satisfy this property on their respective sets of utility functions. However, Proposition S.2 shows that the (societal) headcount ratio violates *Minimal Pareto* on heterogeneous sets of utility functions. Hence, not only is the (societal) headcount ratio not monotonic, but it is not even minimally welfare-consistent.

Proposition S.2.
On any heterogeneous $U \subseteq U^B$, the headcount ratio below the global line z with $X_z = X_Q(U)$ violates Minimal Pareto.

Proof. We construct two distributions $\mathbf{y}, \mathbf{y}' \in Y^3$ and show that for some $\mathbf{u}' \in U^3$ *Minimal Pareto* implies $P_U(\mathbf{y},\mathbf{u}') < P_U(\mathbf{y}',\mathbf{u}')$ but $H_S(\mathbf{y},\mathbf{u}') = H_S(\mathbf{y}',\mathbf{u}')$.

Since U is heterogeneous, there exist two $u, u' \in U$ and some $(y, \overline{y}) \in X$ with $\overline{y} \geq \overline{y}^z$ such that $(y,\overline{y}) \in X_Q(u)$ and $(y,\overline{y}) \notin X_Q(u')$. As $(y,\overline{y}) \notin X_Q(u')$ and $\overline{y} \geq \overline{y}^z$, we must have $y \geq z_a$. If $y = z_a$, then by the continuity of u there exists $y' > y$ such that $(y',\overline{y}) \in X_Q(u)$ and $(y',\overline{y}) \notin X_Q(u')$. Therefore, we can assume without loss of generality that $y > z_a$.

Consider the income distribution $\mathbf{y} := (y, \overline{y}, \overline{y}) \in Y^3$, which is such that $(y_2, \overline{\mathbf{y}}) = (y_3, \overline{\mathbf{y}}) \notin X_Q(U)$. For $\delta = \frac{\overline{y}^z}{\overline{y}}$, we define a second distribution $\mathbf{y}' \in Y^3$ by letting $y_i' = \delta y_i$ for all $i \in \{1,2,3\}$. By construction, we have $\overline{\mathbf{y}}' = \overline{y}^z$.

[7] Indices H_S and HH_S both violate *Weak Pareto*. Index HH_S does not satisfy *Weak Pareto* because its poverty score function is constant for all levels of own income smaller than z_a.

We show that the preconditions for *Minimal Pareto* are met for $\mathbf{u}' = (u', u', u')$. First, we have $u'_i(y_i, \overline{y}) > u'_i(y'_i, \overline{y}')$ for all $i \in \{1, 2, 3\}$ if we have $\delta < 1$. As $(y, \overline{y}) \in X_Q(u)$ and $y > z_a$, we must have $\overline{y} > \overline{y}^z$, which yields $\delta < 1$. Second, we have by construction that $y_i = \frac{\overline{\mathbf{y}}}{\overline{\mathbf{y}}'} y'_i$ for all $i \in \{2, 3\}$. Finally, we show that $1 \in Q(\mathbf{y}', \mathbf{u}')$, $1 \notin Q(\mathbf{y}, \mathbf{u}')$ and $y_1 > z_a$. We have $y_1 > z_a$ because $y_1 = y > z_a$. We have $1 \notin Q(\mathbf{y}, \mathbf{u}')$ because $u'_1 = u'$ and $(y, \overline{y}) \notin X_Q(u')$. There remains to show that $1 \in Q(\mathbf{y}', \mathbf{u}')$. It is sufficient to show that $y'_1 < z_a$ because $\overline{\mathbf{y}}' = \overline{y}^z$. As $y'_1 = y \frac{\overline{y}^z}{\overline{y}}$, we have $y'_1 < z_a$ if we can show $\frac{y}{\overline{y}} < \frac{z_a}{\overline{y}^z}$. As $(y, \overline{y}) \in X_Q(u)$, we have $u(y, \overline{y}) < u(z_a, \overline{y}^z)$, which implies together with $\overline{y} > \overline{y}^z$ that $y < z_a \frac{\overline{y}}{\overline{y}^z}$ because utility functions are strictly increasing in own income when relative income is held constant. We have shown that all the preconditions for *Minimal Pareto* are met and thus $P_U(\mathbf{y}, \mathbf{u}') < P_U(\mathbf{y}', \mathbf{u}')$.

To conclude, we show that $H_S(\mathbf{y}, \mathbf{u}') = H_S(\mathbf{y}', \mathbf{u}') = 1/3$ when $X_z = X_Q(U)$. To do this, it is sufficient to prove that $(y_1, \overline{\mathbf{y}}), (y'_1, \overline{\mathbf{y}}') \in X_z$ and that $(y_2, \overline{\mathbf{y}}), (y_3, \overline{\mathbf{y}})$, $(y'_2, \overline{\mathbf{y}}'), (y'_3, \overline{\mathbf{y}}') \notin X_z$. We have $(y_1, \overline{\mathbf{y}}) \in X_z$ because $(y, \overline{y}) \in X_Q(u)$. We have $(y'_1, \overline{\mathbf{y}}') \in X_z$ because $y'_1 < z_a$ and $\overline{\mathbf{y}}' = \overline{y}^z$. We have $(y_2, \overline{\mathbf{y}}), (y_3, \overline{\mathbf{y}}) \notin X_z$ by Lemma 2 because $y_2 = y_3 \geq \overline{y}$. In order to show $(y'_2, \overline{\mathbf{y}}'), (y'_3, \overline{\mathbf{y}}') \notin X_z$, we show that $y'_2 = y'_3 \geq z_a$ (this is sufficient because $\overline{\mathbf{y}}' = \overline{y}^z$). As $y'_i = \frac{\overline{y}^z}{\overline{y}} y_i$ for all $i \in \{1, 2, 3\}$ and $y_2 = y_3 \geq \overline{y}$, we have $y'_2 = y'_3 \geq \overline{y}^z$. We have $y'_2 = y'_3 \geq z_a$ because we assume $\overline{y}^z \geq z_a$, the desired result. ■

The HH_S satisfies *Minimal Pareto* on any subset of U^C, where U^C is the subset of U^B on which indifference curves are convex.

Proposition S.3.
On any $U \subseteq U^C$, the hierarchical headcount ratio below the global line z with $X_z = X_Q(U)$ satisfies Minimal Pareto.

Proof. Take any $(\mathbf{y}, \mathbf{u}), (\mathbf{y}', \mathbf{u}) \in \mathcal{X}_{U^C}$ that satisfy the preconditions under which *Minimal Pareto* implies $P_U(\mathbf{y}', \mathbf{u}) < P_U(\mathbf{y}, \mathbf{u})$. These preconditions require that $u_i(y'_i, \overline{y}') \geq u_i(y_i, \overline{y})$ for all $i \in N(\mathbf{y})$. By Lemma 1, this implies in turn that $\overline{y} \leq \overline{y}'$.

First, we show that $p^{HH_S}(y'_i, \overline{y}') \leq p^{HH_S}(y_i, \overline{y})$ for all $i \in N(\mathbf{y}) = N(\mathbf{y}')$.

For all $i \in Q(\mathbf{y}, \mathbf{u})$ for whom $(y_i, \overline{y}) \in X_A$, the two inequalities $\overline{y} \leq \overline{y}'$ and $u_i(y'_i, \overline{y}') \geq u_i(y_i, \overline{y})$ imply that $y'_i \geq y_i$. This implies that $p^{HH_S}(y'_i, \overline{y}') \leq p^{HH_S}(y_i, \overline{y})$.

For all $i \in Q(\mathbf{y}, \mathbf{u})$ for whom $(y_i, \overline{y}) \in X_z \backslash X_A$, the convexity of the indifference curves of u_i together with the inequalities $\overline{y} \leq \overline{y}'$, $u_i(y'_i, \overline{y}') \geq u_i(y_i, \overline{y})$ and

$u_i(z_a, \overline{y}^z) > u_i(y_i, \overline{y})$ imply that[8]

$$y_i' \geq z_a + \frac{y_i - z_a}{\overline{y} - \overline{y}^z}(\overline{y}' - \overline{y}^z).$$

Last inequality can be rewritten $\frac{y_i'-z_a}{\overline{y}'-\overline{y}^z} \geq \frac{y_i-z_a}{\overline{y}-\overline{y}^z}$, which by the definition of p implies that $p^{HH_S}(y_i', \overline{y}') \leq p^{HH_S}(y_i, \overline{y})$.

For all $i \notin Q(\mathbf{y}, \mathbf{u})$, the preconditions require that $y_i' \geq \frac{\overline{y}'}{\overline{y}} y_i$, where $\frac{\overline{y}'}{\overline{y}} \geq 1$. This requirement directly implies that $p^{HH_S}(y_i', \overline{y}') \leq p^{HH_S}(y_i, \overline{y})$ if $y_i < z_a$. For the case $y_i \geq z_a$, this requirement also implies $p^{HH_S}(y_i', \overline{y}') \leq p^{HH_S}(y_i, \overline{y})$ under the assumption that $z(\overline{y}') \leq \frac{\overline{y}'}{\overline{y}} z(\overline{y})$. There remains to derive a contradiction when assuming that $z(\overline{y}') > \frac{\overline{y}'}{\overline{y}} z(\overline{y})$. As $\overline{y}' \geq \overline{y}$, this contradiction assumption implies that $\overline{y}' > \overline{y}$ and thus $\frac{\overline{y}'}{\overline{y}} > 1$. As $X_z = X_Q(U)$, this contradiction assumption implies that there exists a utility function $u \in U$ such that $(\frac{\overline{y}'}{\overline{y}} z(\overline{y}), \overline{y}') \in X_Q(u)$. As by definition of $z(\overline{y})$ there exists no $u' \in U$ for which $(z(\overline{y}), \overline{y}) \in X_Q(u')$, we must have $(z(\overline{y}), \overline{y}) \notin X_Q(u)$. Therefore we have $u(z(\overline{y}), \overline{y}) \geq u(\frac{\overline{y}'}{\overline{y}} z(\overline{y}), \overline{y}')$. This is equivalent to $u(z(\overline{y}), \overline{y}) \geq u(\frac{\overline{y}'}{\overline{y}} z(\overline{y}), \frac{\overline{y}'}{\overline{y}} \overline{y})$, a contradiction to utility function being strictly increasing in own income when relative income is kept constant.

Second, the preconditions also require that $y_\ell' > z_a$ and $u_\ell(y_\ell', \overline{y}') \geq u_\ell(z_a, \overline{y}^z)$ for some $\ell \in Q(\mathbf{y}, \mathbf{u})$. As $\ell \in Q(\mathbf{y}, \mathbf{u})$, we must have $(y_\ell, \overline{y}) \in X_z$ because $X_z = X_Q(U)$. For the case $(y_\ell, \overline{y}) \in X_A$, we have $p^{HH_S}(y_\ell, \overline{y}) = 1$. Since $y_\ell' > z_a$, we directly get $p^{HH_S}(y_\ell', \overline{y}') < 1$, and so $p^{HH_S}(y_\ell', \overline{y}') < p^{HH_S}(y_\ell, \overline{y})$. For the alternative case $(y_\ell, \overline{y}) \in X_z \backslash X_A$, the argument used above (for the case $i \in Q(\mathbf{y}, \mathbf{u})$ for whom $(y_i, \overline{y}) \in X_z \backslash X_A$) yields $\frac{y_\ell'-z_a}{\overline{y}'-\overline{y}^z} > \frac{y_\ell-z_a}{\overline{y}-\overline{y}^z}$, and therefore $p^{HH_S}(y_\ell', \overline{y}') < p^{HH_S}(y_\ell, \overline{y})$. In both cases, we obtain $HH_S(\mathbf{y}') < HH_S(\mathbf{y})$, the desired result. ■

[8] Assume to the contrary that $y_i' < z_a + \frac{y_i-z_a}{\overline{y}-\overline{y}^z}(\overline{y}' - \overline{y}^z)$. The inequality $u_i(y_i', \overline{y}') \geq u_i(y_i, \overline{y})$ and the convexity of u_i imply that

$$u_i(y_i, \overline{y}) \geq u_i\left(y_i' - \frac{y_i' - y_i}{\overline{y}' - \overline{y}}(\overline{y}' - \overline{y}^z), \overline{y}^z\right).$$

Then, the inequality $y_i' < z_a + \frac{y_i-z_a}{\overline{y}-\overline{y}^z}(\overline{y}' - \overline{y}^z)$ implies that

$$u_i(y_i, \overline{y}) \geq u_i\left(z_a, \overline{y}^z\right),$$

showing that $i \notin Q(\mathbf{y}, \mathbf{u})$, a contradiction.

S4 Additional Tables and Figures

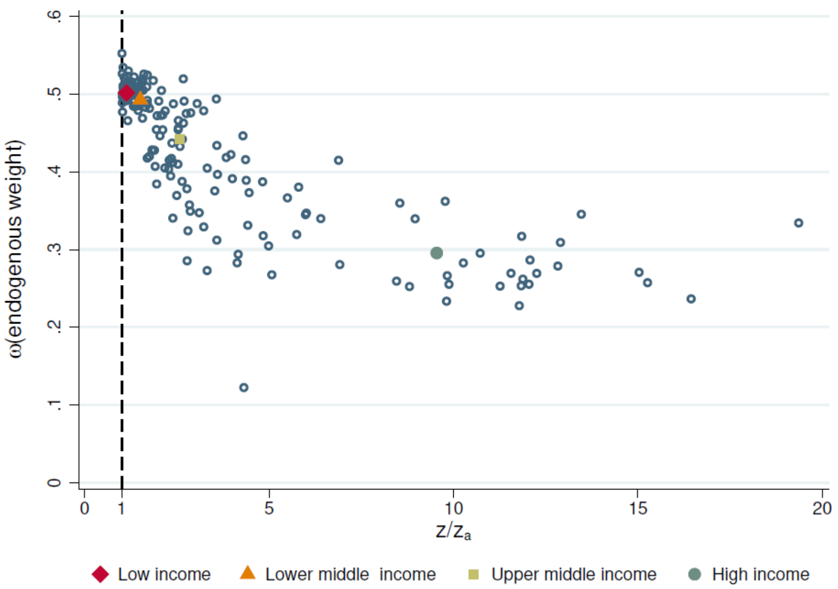


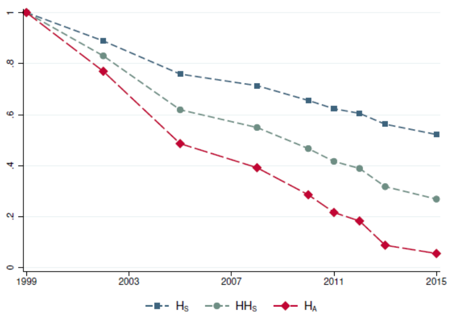
Figure S.1: Endogenous weight as a function of the ratio of global lines

Notes: The graph plots the median endogenous weight ω *as a function of the median ratio between the weakly relative line and the absolute line by country. We only consider country-year pairs with* $z(\overline{y}) \geq z_a$*. The following countries with a median ratio of global lines below 1 are excluded: Burundi, Central African Republic, Democratic Republic of Congo, Madagascar, Malawi, Mozambique, and Rwanda. Full markers display the median values within each country income group as defined by the World Bank.*

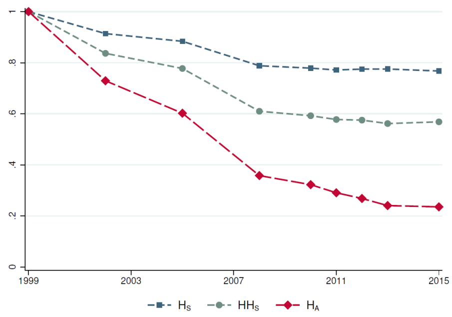
Table S.1: Poverty by region, 1999 & 2015

	1999					2015				
	H_S	HH_S	H_A	Mean inc. (PPP$)	Pop. (million)	H_S	HH_S	H_A	Mean inc. (PPP$)	Pop. (million)
East Asia & Pacific	43.4	38.3	33.8	210	1977	22.7	10.3	1.9	421	2223
Europe & Central Asia	20.5	10.7	4.5	692	858	15.8	6.1	1	935	906
Latin America & Caribbean	34.4	24.1	13.9	324	493	26.7	14.1	3.7	496	572
Middle East & North Africa	28.3	14.8	3.6	274	279	22.6	11.2	4.1	340	376
North America	18.8	6.8	.7	1838	309	18.4	6.8	1.1	1995	356
South Asia	51.2	46.1	40.7	83	1345	33	22.8	13.2	128	1715
Sub-Saharan Africa	62.3	60.5	58.7	81	631	49.7	46.1	42	108	991
World	41.1	34.5	29	335	5892	28.1	17.9	10.3	453	7139

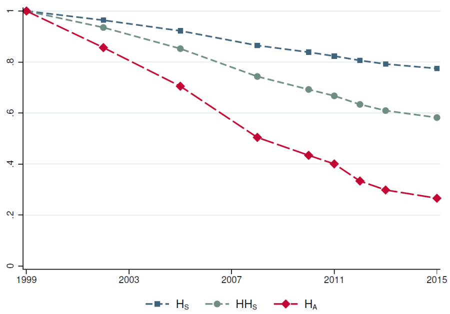
Notes: Mean income per capita is expressed in PPP$ per month. Regions as defined by the World Bank.



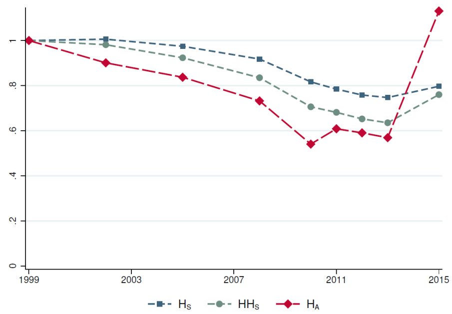
(a) East Asia and Pacific



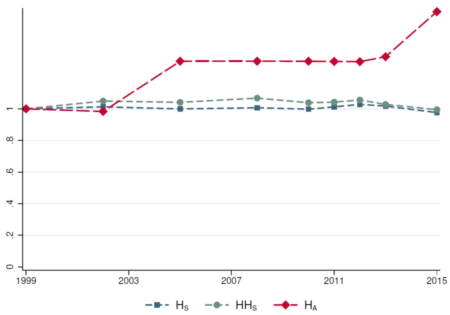
(b) Europe and Central Asia



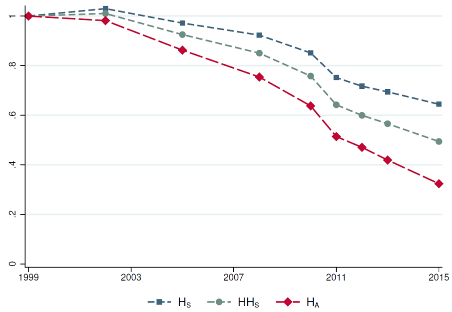
(c) Latin America and the Caribbean



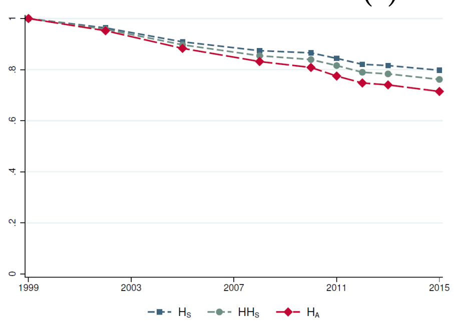
(d) Middle East and North Africa



(e) North America



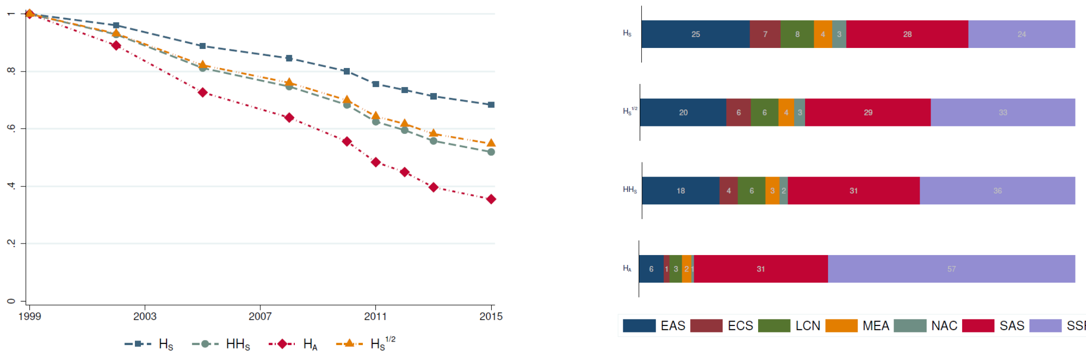
(f) South Asia



(g) Sub-Saharan Africa

Figure S.2: Evolution of poverty by region, 1990-2015

Notes: The graph plots the evolution of poverty relative to 1999 for all reference years through 2015. Each region includes all countries with available information in each reference year.



(a) Evolution of global poverty, 1999-2015

(b) Distribution by region, 2015

Figure S.3: Evolution and distribution of global poverty

Notes: Panel a plots the evolution of poverty relative to 1999 for all reference years through 2015. It includes all countries with available information in each reference year. Panel b plots the contribution to global poverty for each of the following regions: East Asia & Pacific (EAS), Europe & Central Asia (ECS), Latin America & Caribbean (LCN), Middle East & North Africa (MEA), North America (NAC), South Asia (SAS), and Sub-Saharan Africa (SSF).

S5 Defining Relative Income Using Mean Income

Although we assume throughout that relative income is defined with respect to median income, all our results also hold when relative income is defined with respect to mean income. That is, our results also hold when $\overline{y}$ denotes mean income in the distribution.

The formal proofs for these claims can be found in our discussion paper, where we prove our results for both definitions of $\overline{y}$. In this section, we do not repeat all proofs when assuming that $\overline{y}$ denotes mean income. Rather, we do two things. First, we explain how to adapt the typical argument that deals with the other-regarding aspect of individual preferences. Second, we provide the argument that is the most challenging to adapt when $\overline{y}$ denotes mean income. This argument is used in Step 2 of the proof that only fair additive indices satisfy *Domination* (Proposition 1).

S5.1 Dealing with ORP with Respect to Mean Income

Our results show how our axioms constrain the trade-off that the measure makes between own income and relative income. That is, they show how the poverty score function compares the bundles of individuals living in societies with different values for $\overline{y}$. The difficulty is that our axioms do not constrain the comparison of bundles, but rather the comparison of income distributions.[9] Hence, our axioms do not constrain how the poverty measure compares two bundles $(y_i, \overline{y})$ and $(y'_i, \overline{y}')$, but rather two distributions $\mathbf{y}$ and $\mathbf{y}'$. Yet, the problem vanishes when the bundles $(y_j, \overline{y})$ and $(y'_j, \overline{y}')$ for all $j \neq i$ are attributed a zero poverty score by function p.

When $\overline{y}$ denotes median income, the difficulty is easily circumvented when considering distributions $\mathbf{y}$ and $\mathbf{y}'$ for which all other individuals earn an income equal to the median income, i.e., when $y_j = \overline{y}$ and $y'_j = \overline{y}'$ for all $j \neq i$. Indeed, such definition will be such that for all $(y_i, \overline{y}), (y'_i, \overline{y}') \in X_Q(U)$ for which $\overline{y}' \geq \overline{y}$ we have that these distributions are such that

- $\overline{\mathbf{y}} = \overline{y}$ and $\overline{\mathbf{y}}' = \overline{y}'$,
- $(y_j, \overline{y}), (y'_j, \overline{y}') \notin X_Q(U)$ and thus $p(y_j, \overline{y}) = p(y'_j, \overline{y}') = 0$,
- $y'_j \geq \frac{\overline{y}'}{\overline{y}} y_j$,
- $u_j(y'_j, \overline{y}') > u_j(y_j, \overline{y})$ for all $u_j \in U^B$ because utility is strictly increasing in own income when relative income is kept constant,

which shows that the precondition for our axioms are met.

As Lemma S.1 shows, one can also find distributions with the required properties when $\overline{y}$ denotes mean income. We consider here the case of two bundles $(y, \overline{y}), (y', \overline{y}') \in X_Q(U)$ with $\overline{y}' \geq \overline{y}$ such that $y' \leq \frac{\overline{y}'}{\overline{y}} y$, because this is the relevant case for which there exist two preferences $u, u' \in U^B$ that disagree on the comparison of these bundles, i.e., $u(y, \overline{y}) > u(y', \overline{y}')$ and $u'(y, \overline{y}) \leq u'(y', \overline{y}')$.[10]

Lemma S.1.
Consider any $U \subseteq U^B$ and any $(y, \overline{y}), (y', \overline{y}') \in X_Q(U)$ such that $\overline{y}' \geq \overline{y}$ and $y' \leq \frac{\overline{y}'}{\overline{y}} y$. There exist distributions $\mathbf{y}, \mathbf{y}' \in Y^n$ such that (i) $\overline{\mathbf{y}} = \overline{y}$ and $\overline{\mathbf{y}}' = \overline{y}'$, (ii) $(y_1, \overline{y}) = (y, \overline{y})$, $(y'_1, \overline{y}') = (y', \overline{y}')$, (iii) $(y_j, \overline{y}), (y'_j, \overline{y}') \notin X_Q(U)$ for all $j \neq i$, (iv) $y'_j \geq \frac{\overline{y}'}{\overline{y}} y_j$ and (v) $u_j(y'_j, \overline{y}') > u_j(y_j, \overline{y})$ for all $j \neq i$ and all $u_j \in U^B$.

Proof. We show that when $(y_1, \overline{y}) := (y, \overline{y})$, $(y'_1, \overline{y}') := (y', \overline{y}')$ and

$$\begin{aligned} y_j &:= \overline{y} + (\overline{y} - y_1)/(n-1), \\ y'_j &:= \overline{y}' + (\overline{y}' - y'_1)/(n-1), \end{aligned}$$

for all $j \neq 1$, the distributions $\mathbf{y}, \mathbf{y}' \in Y^n$ have the desired properties.

[9] More precisely, they constrain the comparison of distribution-profile pairs.

[10] Additional arguments are required in order to extend Lemma S.1 for the comparison of bundles $(y, \overline{y})$ and $(y', \overline{y}')$ for which $u(y, \overline{y}) < u(y', \overline{y}')$ for all $u \in U^B$.

Part (i) is immediate given that $n\overline{y} = \sum_i y_i$ when $\overline{y}$ denotes mean income.

Part (ii) is by construction.

Part (iii) requires that $(y_j, \overline{y}) \notin X_Q(U)$. By Lemma 2, it is sufficient to show that $y_j \geq \overline{y}$. As $(y_1, \overline{y}) \in X_Q(U)$, we have that $y_1 < \overline{y}$ (Lemma 2). As all $j \neq 1$ earn the same income and $y_1 < \overline{y}$, we have that $n\overline{y} = \sum_i y_i$ only when $y_j \geq \overline{y}$, as desired.

Part (iv) requires that $y'_j \geq \frac{\overline{y}'}{\overline{y}} y_j$. By construction, we have that $y'_1 \leq \frac{\overline{y}'}{\overline{y}} y_1$ because $y' \leq \frac{\overline{y}'}{\overline{y}} y$. As all $j \neq 1$ earn the same income y_j in $\mathbf{y}$ and y'_j in $\mathbf{y}'$, we have that $\sum_i y'_i = \frac{\overline{y}'}{\overline{y}} \sum_i y_i$ only if $y'_j \geq \frac{\overline{y}'}{\overline{y}} y_j$ because $y'_1 \leq \frac{\overline{y}'}{\overline{y}} y_1$, as desired.

Part (v) is a direct consequence of the fact that $y'_j \geq \frac{\overline{y}'}{\overline{y}} y_j$ because utility is strictly increasing in own income when relative income is kept constant. ■

S5.2 Adapting the Proof for Proposition 1

We show how to adapt the key argument provided in Step 2 of Proposition 1.

Step 2. P_U satisfies Domination only if for all $(\mathbf{y}, \mathbf{u}) \in \mathcal{X}_U$

$$P_U(\mathbf{y}, \mathbf{u}) = p(z_a, z_a) + \frac{1}{n(\mathbf{y})} \sum_{i \in Q^*(\mathbf{y})} p'(y_i, \overline{y}) \tag{S.1}$$

where $Q^*(\mathbf{y}) = \{i \in N(\mathbf{y}) | (y_i, \overline{y}) \in X_Q(U)\}$ and $p' : X \to \mathbb{R}$ is well-defined on X and continuous on $X_Q(U)$.

Recall the definition of the supremum function $\hat{z} : [z_a, \infty) \to \mathbb{R}_+$

$$\hat{z}(\overline{y}) := \sup\{y \geq 0 | u(z_a, \overline{y}^z) = u(y, \overline{y}) \text{ for some } u \in U\},$$

which is such that $X_{\hat{z}} = X_Q(U)$ because $(y, \overline{y}) \in X_Q(U)$ if and only if $y < \hat{z}(\overline{y})$.

We do not repeat the argument showing that $p(\overline{y}, \overline{y}) = p(z_a, z_a)$ for all $\overline{y} \geq z_a$. Rather, we only establish Eq. (S.1) for the case for which $\overline{y}$ denotes mean income.

We first show, for all $(y, \overline{y}) \in X \backslash X_Q(U)$, that

$$p(y, \overline{y}) = \alpha(\overline{y})(y - \overline{y}) + p(z_a, z_a) \tag{S.2}$$

for some continuous function $\alpha : [z_a, \infty) \to \mathbb{R}$.

We start by showing Eq. (S.2) for an arbitrary but fixed level of mean income $\overline{y} \in [z_a, \infty)$. We derive a contradiction if there exists no $\hat{\alpha} = \alpha(\overline{y}) \in \mathbb{R}$ such that Eq. (S.2) holds. Under this contradiction assumption, there must exist $\delta > 0$ and two income levels $y, y' \in [\hat{z}(\overline{y}), \infty)$ for which[11]

$$p(y+\delta, \overline{y}) - p(y, \overline{y}) \neq p(y'+\delta, \overline{y}) - p(y', \overline{y}). \tag{S.3}$$

Consider any two distributions $\mathbf{y}, \mathbf{y}' \in Y^n$ such that $\overline{\mathbf{y}} = \overline{\mathbf{y}}' = \overline{y}$, $y_1 = y+\delta$, $y_2 = y'$, $y'_1 = y$, $y'_2 = y'+\delta$, and $y_j = y'_j$ for all $j \in \{3, \ldots, n\}$.[12] Since $y \geq \hat{z}(\overline{y})$, $y' \geq \hat{z}(\overline{y})$ and $\delta > 0$, bundles $(y, \overline{y})$, $(y+\delta, \overline{y})$, $(y', \overline{y})$, $(y'+\delta, \overline{y})$ are all in $X \backslash X_Q(U)$. Therefore, *Domination* implies for all $\mathbf{u} \in U^n$ that $P_U(\mathbf{y}, \mathbf{u}) = P_U(\mathbf{y}', \mathbf{u})$. Using Eq. (S.5), we get

$$p(y+\delta, \overline{y}) + p(y', \overline{y}) = p(y'+\delta, \overline{y}) + p(y, \overline{y}),$$

a contradiction to Eq. (S.3).

There remains to show that function α is continuous. Assume to the contrary that α is discontinuous at some level of mean income $\overline{y}^* \in [z_a, \infty)$. For the two income levels $y^p := \hat{z}(\overline{y}^*)/2$ and $y^r := 2\overline{y}^* - y^p$, which are such that $y^p < \hat{z}(\overline{y}^*)$ and $y^r > \overline{y}^*$, consider the income distribution $\mathbf{y} := (y^p, y^r)$. By construction, we have $\overline{\mathbf{y}} = \overline{y}^*$. First, consider the case for which $p(y^r, \overline{y}^*) < \lim_{\epsilon \to 0} p(y^r, \overline{y}^* + \epsilon)$ for $\epsilon > 0$. For some $\gamma > 0$, consider the income distribution $\mathbf{y}^\gamma := (y^p + \gamma, y^r)$, which is such that $\overline{\mathbf{y}}^\gamma = \overline{y}^* + \gamma/2$. For γ small enough, we have bundle $(y^p + \gamma, \overline{y}^* + \gamma/2)$ in $X_Q(U)$ and bundle $(y^r, \overline{y}^* + \gamma/2)$ in $X \backslash X_Q(U)$. Also, the monotonicity properties of utility functions imply that $u(y^p + \gamma, \overline{y}^* + \gamma/2) > u(y^p, \overline{y}^*)$ for all $u \in U^B$ because the relative income of the former bundle is larger. Therefore, *Domination* implies for all $\mathbf{u} \in U^2$ that $P_U(\mathbf{y}, \mathbf{u}) \geq P_U(\mathbf{y}^\gamma, \mathbf{u})$ for all $\gamma > 0$ sufficiently small, which by Eq. (S.5) means

$$p(y^p, \overline{y}^*) + p(y^r, \overline{y}^*) \geq p(y^p + \gamma, \overline{y}^* + \gamma/2) + p(y^r, \overline{y}^* + \gamma/2). \tag{S.4}$$

However, since $(y^p, \overline{y}^*), (y^p + \gamma, \overline{y}^* + \gamma/2) \in X_Q(U)$, the continuity of function p on $X_Q(U)$ implies that $\lim_{\gamma \to 0} p(y^p + \gamma, \overline{y}^* + \gamma/2) = p(y^p, \overline{y}^*)$. This leads to a contradiction to Eq. (S.4) because this case is such that $p(y^r, \overline{y}^*) < \lim_{\gamma \to 0} p(y^r, \overline{y}^* + \gamma/2)$. The proof for the alternative case $p(y^r, \overline{y}^*) > \lim_{\epsilon \to 0} p(y^r, \overline{y}^* + \epsilon)$ for $\epsilon > 0$ is based on another distribution $\mathbf{y}^{\gamma'}$ constructed as $\mathbf{y}^{\gamma'} := (y^p - \gamma, y^r + 2\gamma)$. As the reasoning is very similar (the main difference being the direction of inequalities) we do not develop this case. Finally, the cases for which the discontinuity of function α at

[11] Any income level $y'' < \hat{z}(\overline{y})$ is such that $(y'', \overline{y}) \notin X \backslash X_Q(U)$.

[12] Observe that, for all values of y, y' and δ, there exists a value for n sufficiently large to ensure that $y_j = y'_j \geq 0$.

$\overline{y}^*$ comes "from the left" use the same reasoning. This proves that α is continuous, and Eq. (S.2) holds.

We use Eq. (S.2) in order to prove Eq. (S.1). Let $p' : X \to \mathbb{R}$ be defined for all $(y, \overline{y}) \in X$ as

$$p'(y, \overline{y}) := p(y, \overline{y}) - \big(\alpha(\overline{y})(y - \overline{y}) + p(z_a, z_a)\big),$$

where function p' is continuous on $X_Q(U)$ because function α is continuous and function p is continuous on $X_Q(U)$. Letting $P'_U(\mathbf{y}, \mathbf{u}) := \sum_{i=1}^{n(\mathbf{y})} p'(y, \overline{y})$, we get from the definition of p' that $P'_U(\mathbf{y}, \mathbf{u}) = P_U(\mathbf{y}, \mathbf{u}) - p(z_a, z_a)$ because $\sum_i (y_i - \overline{y}) = 0$ when $\overline{y}$ denotes mean income. The definition of p' together with Eq. (S.2) implies that $p'(y, \overline{y}) = 0$ for all $(y, \overline{y}) \in X \backslash X_Q(U)$, which shows that $p'(y_i, \overline{y}) = 0$ for all $i \in N(\mathbf{y}) \backslash Q^*(\mathbf{y})$. Together, we obtain Eq. (S.1).

S6 Proof of Proposition 1

In this section, we show that an additive index satisfies *Domination* only if it is a fair additive index.

Take any additive index $P_U : \mathcal{X}_U \to \mathbb{R}$.

The proof of Proposition 1 has three steps. In Step 1, we show that index P_U satisfies *Domination* only if it is independent of the particular preference profile $\mathbf{u}$. In Step 2, we show that index P_U satisfies *Domination* only if its poverty score function is constant on $X \backslash X_Q(U)$ and weakly decreasing in own income on $X_Q(U)$. In Step 3, we construct a particular function z and show that P_U satisfies *Domination* only if it satisfies the definition of a fair additive index.

Step 1. P_U satisfies Domination only if for all $(\mathbf{y}, \mathbf{u}) \in \mathcal{X}_U$

$$P_U(\mathbf{y}, \mathbf{u}) = \frac{1}{n(\mathbf{y})} \sum_{i=1}^{n(\mathbf{y})} p(y_i, \overline{y}), \tag{S.5}$$

where function $p : X \to \mathbb{R}$ is well-defined on X and continuous on $X_Q(U)$.

Being an additive index, P_U is defined as

$$P_U(\mathbf{y}, \mathbf{u}) = \frac{1}{n(\mathbf{y})} \sum_{i=1}^{n(\mathbf{y})} p_{u_i}(y_i, \overline{y}), \tag{S.6}$$

where for every $u \in U$, $p_u : X \to \mathbb{R}$ is well-defined on X and continuous on $X_Q(u)$.

First, we show that the poverty score function $p : X \times U \to \mathbb{R}$ such that $p : (y, \overline{y}, u) \mapsto p_u(y, \overline{y})$ is independent of u. Suppose, to the contrary, that there exist $u, u' \in U$ such that, for some $(y, \overline{y}) \in X$, $p_u(y, \overline{y}) \neq p_{u'}(y, \overline{y})$. It is possible to construct two pairs $(\mathbf{y}, \mathbf{u}), (\mathbf{y}, \mathbf{u}') \in \mathcal{X}_U$ such that $\mathbf{y}$ has $y_1 = y$ and $\overline{\mathbf{y}} = \overline{y}$ and such that $u_1 = u$, $u'_1 = u'$ and $u_j = u'_j$ for all $j \in N(\mathbf{y}) \setminus \{1\}$. By Eq. (S.6), we have $P_U(\mathbf{y}, \mathbf{u}) \neq P_U(\mathbf{y}, \mathbf{u}')$ because by construction $p_{u_1}(y_1, \overline{y}) \neq p_{u'_1}(y_1, \overline{y})$ and $p_{u_j}(y_j, \overline{y}) = p_{u'_j}(y_j, \overline{y})$ for all $j \in N(\mathbf{y}) \setminus \{1\}$. As the two distribution-profile pairs feature the same distribution $\mathbf{y}$, *Domination* implies that both $P_U(\mathbf{y}, \mathbf{u}') \leq P_U(\mathbf{y}, \mathbf{u})$ and $P_U(\mathbf{y}, \mathbf{u}') \geq P_U(\mathbf{y}, \mathbf{u})$. Therefore we have $P_U(\mathbf{y}, \mathbf{u}) = P_U(\mathbf{y}, \mathbf{u}')$, yielding the desired contradiction. We have thus shown that P_U is based on a degenerate poverty score function $p : X \to \mathbb{R}$, which is well-defined on X.

Second, we show that the function $p : X \to \mathbb{R}$ is continuous on $X_Q(U)$. For any bundle $(y, \overline{y}) \in X_Q(U)$ there exists some $u \in U$ such that $(y, \overline{y}) \in X_Q(u)$ (by the definition of $X_Q(U)$). By the definition of an additive index, p_u is continuous on $X_Q(u)$. Thus p is continuous at $(y, \overline{y})$ because $(y, \overline{y}) \in X_Q(u)$ and $p = p_u$. This concludes the proof for Step 1.

Step 2. P_U satisfies Domination only if for all $(\mathbf{y}, \mathbf{u}) \in \mathcal{X}_U$

$$P_U(\mathbf{y}, \mathbf{u}) = p(z_a, z_a) + \frac{1}{n(\mathbf{y})} \sum_{i \in Q^*(\mathbf{y})} p'(y_i, \overline{y}) \tag{S.7}$$

where $Q^*(\mathbf{y}) := \{i \in N(\mathbf{y}) | (y_i, \overline{y}) \in X_Q(U)\}$ and $p' : X \to \mathbb{R}$ is well-defined on X and continuous on $X_Q(U)$. Moreover, p' is weakly decreasing in its first argument on $X_Q(U)$ and $p'(y_i, \overline{y}) \geq 0$ for all $i \in Q^*(\mathbf{y})$.

First, we establish Eq. (S.7). For any $(\mathbf{y}, \mathbf{u}) \in \mathcal{X}_U$, Eq. (S.5) implies

$$\begin{aligned} P_U(\mathbf{y}, \mathbf{u}) &= \frac{1}{n(\mathbf{y})} \sum_{i=1}^{n(\mathbf{y})} p(y_i, \overline{y}) \\ &= p(z_a, z_a) + \frac{1}{n(\mathbf{y})} \sum_{i=1}^{n(\mathbf{y})} \left(p(y_i, \overline{y}) - p(z_a, z_a) \right). \end{aligned}$$

Let function p' be defined as $p'(y, \overline{y}) := p(y, \overline{y}) - p(z_a, z_a)$ for all $(y, \overline{y}) \in X$. Function p' inherits continuity on $X_Q(U)$ from function p. If $p'(y, \overline{y}) = 0$ for all

$(y, \overline{y}) \in X \backslash X_Q(U)$, then we would obtain

$$P_U(\mathbf{y}, \mathbf{u}) = p(z_a, z_a) + \frac{1}{n(\mathbf{y})} \sum_{i \in Q^*(\mathbf{y})} p'(y_i, \overline{y})$$

because $Q^*(\mathbf{y}) = \{i \in N(\mathbf{y}) | (y_i, \overline{y}) \in X_Q(U)\}$.

In order to prove that $p'(y, \overline{y}) = 0$ for all $(y, \overline{y}) \in X \backslash X_Q(U)$, we show that $p(y, \overline{y}) = p(z_a, z_a)$ for any $(y, \overline{y}) \in X \backslash X_Q(U)$. Consider the two distributions $\mathbf{y}, \mathbf{y}' \in Y^3$ defined as $\mathbf{y} := (y, \overline{y}, \overline{y})$ and $\mathbf{y}' := (\overline{y}, \overline{y}, \overline{y})$. Note that $\overline{\mathbf{y}} = \overline{\mathbf{y}}' = \overline{y}$. We show that $(y_i, \overline{y}), (y'_i, \overline{y}') \notin X_Q(u)$ for all $i \in \{1, 2, 3\}$ and all $u \in U$. This is immediate for $(y_1, \overline{y}) = (y, \overline{y})$ because we assumed $(y, \overline{y}) \in X \backslash X_Q(U)$. All other bundles correspond to $(\overline{y}, \overline{y})$, which by Lemma 2 is such that $(\overline{y}, \overline{y}) \notin X_Q(U)$ because $\overline{y} \geq z_a$. As $(y_i, \overline{y}), (y'_i, \overline{y}') \notin X_Q(u)$ for all $i \in \{1, 2, 3\}$ and all $u \in U$, *Domination* implies $P_U(\mathbf{y}, \mathbf{u}) \leq P_U(\mathbf{y}', \mathbf{u})$ and $P_U(\mathbf{y}, \mathbf{u}) \geq P_U(\mathbf{y}', \mathbf{u})$ for all $\mathbf{u} \in U^3$. Therefore, we have $P_U(\mathbf{y}, \mathbf{u}) = P_U(\mathbf{y}', \mathbf{u})$. By Eq. (S.5), we also have that $P_U(\mathbf{y}, \mathbf{u}) = \frac{1}{3} p(y, \overline{y}) + \frac{2}{3} p(\overline{y}, \overline{y})$ and $P_U(\mathbf{y}', \mathbf{u}) = p(\overline{y}, \overline{y})$ for all $\mathbf{u} \in U^3$. Therefore, we have $p(y, \overline{y}) = p(\overline{y}, \overline{y})$ for any $(y, \overline{y}) \in X \backslash X_Q(U)$ because $P_U(\mathbf{y}, \mathbf{u}) = P_U(\mathbf{y}', \mathbf{u})$.

Thus, Eq. (S.7) holds if we can show that $p(\overline{y}, \overline{y}) = p(z_a, z_a)$ for all $\overline{y} \geq z_a$. For any $\overline{y} \geq z_a$ and for any $n \in \mathbb{N}$, consider the distributions $\mathbf{z}_a := (z_a, \ldots, z_a) \in Y^n$ and $\mathbf{y} := (\overline{y}, \ldots, \overline{y}) \in Y^n$. By construction, we have $\overline{\mathbf{z}}_a = z_a$ and $\overline{\mathbf{y}} = \overline{y}$. By Lemma 2, $(z_a, z_a), (\overline{y}, \overline{y}) \notin X_Q(U)$, which by definition implies that $(z_a, z_a), (\overline{y}, \overline{y}) \notin X_Q(u)$ for all $u \in U$. It follows from *Domination* that $P_U(\mathbf{z}_a, \mathbf{u}) \leq P_U(\mathbf{y}, \mathbf{u})$ and $P_U(\mathbf{z}_a, \mathbf{u}) \geq P_U(\mathbf{y}, \mathbf{u})$, so $P_U(\mathbf{z}_a, \mathbf{u}) = P_U(\mathbf{y}, \mathbf{u})$ for all $\mathbf{u} \in U^n$. But by Eq. (S.5), $P_U(\mathbf{z}_a, \mathbf{u}) = p(z_a, z_a)$ and $P_U(\mathbf{y}, \mathbf{u}) = p(\overline{y}, \overline{y})$, so $p(\overline{y}, \overline{y}) = p(z_a, z_a)$ as required.

Second, we show that function p' is weakly decreasing in its first argument on $X_Q(U)$ and that $p'(y_i, \overline{y}) \geq 0$ for all $i \in Q^*(\mathbf{y})$.

Let the "supremum" function $\hat{z} : [z_a, \infty) \to \mathbb{R}_+$ be defined as

$$\hat{z}(\overline{y}) := \sup\{y \geq 0 | u(z_a, \overline{y}^z) = u(y, \overline{y}) \text{ for some } u \in U\}.$$

The definition of function $\hat{z}$ is such that for any $(y, \overline{y}) \in X$ we have $y < \hat{z}(\overline{y})$ if and only if $(y, \overline{y}) \in X_Q(U)$. In our notation, this means that $X_{\hat{z}} = X_Q(U)$. Therefore, we have $i \in Q^*(\mathbf{y})$ if and only if $y_i < \hat{z}(\overline{y})$.

We start by showing that function p' is weakly decreasing in its first argument on $X_Q(U)$. We must thus show that $p'(y, \overline{y}) \geq p'(y', \overline{y})$ for all $(y', \overline{y}) \in X_Q(U)$ for which $y' \in [y, \hat{z}(\overline{y}))$. Consider the two distributions $\mathbf{y}, \mathbf{y}' \in Y^3$ defined as $\mathbf{y} := (y, \overline{y}, \overline{y})$ and $\mathbf{y}' := (y', \overline{y}, \overline{y})$. By construction, we have $\overline{\mathbf{y}} = \overline{\mathbf{y}}' = \overline{y}$. By

Lemma 2, we have that $(y_2, \overline{y}), (y_2', \overline{y}'), (y_3, \overline{y}), (y_3', \overline{y}') \notin X_Q(U)$. We also have $u(y_1, \overline{y}) < u(y_1', \overline{y}')$ for all $u \in U$ because $y_1 < y_1'$ and $\overline{y} = \overline{y}'$. Thus, *Domination* implies that $P_U(\mathbf{y}, \mathbf{u}) \geq P_U(\mathbf{y}', \mathbf{u})$ for all $\mathbf{u}$. Therefore, Eq. (S.7) implies that $p'(y, \overline{y}) \geq p'(y', \overline{y})$, as desired.

We now show that $p'(y_i, \overline{y}) \geq 0$ for all $i \in Q^*(\mathbf{y})$. We must thus show that $p'(y_i, \overline{y}) \geq 0$ for all $i \in N(\mathbf{y})$ for whom $y_i < \hat{z}(\overline{y})$. We have shown above that $p'(y, \overline{y}) = 0$ when $(y, \overline{y}) \notin X_Q(U)$, i.e., when $y \geq \hat{z}(\overline{y})$. There remains to show that $p'(y_i, \overline{y}) \geq p'(y, \overline{y})$ when $y_i < \hat{z}(\overline{y}) \leq y$. This is implied by the reasoning used in the previous paragraph when modifying the definitions of distributions $\mathbf{y}$ and $\mathbf{y}'$ in such a way that $y_1 = y_i$ and $y_1' = y$, the desired result. This concludes the proof for Step 2.

Step 3. P_U satisfies Domination only if it satisfies the definition of a fair additive index.

We construct a continuous function $z : [z_a, \infty) \to \mathbb{R}_+$ for which function p' defined in Step 2 satisfies the definition of a fair additive index. This function z is defined using function $\hat{z}$ and function p' as follows

$$z(\overline{y}) := \min\{y \in [0, \hat{z}(\overline{y})] | p'(y, \overline{y}) = 0\}.$$

We show that function z is well-defined. Recall that $p'(y, \overline{y}) = 0$ for all $(y, \overline{y}) \notin X_Q(U)$. We therefore have $p'(\hat{z}(\overline{y}), \overline{y}) = 0$ for all $\overline{y} \geq z_a$ because the definition of function $\hat{z}$ is such that $(\hat{z}(\overline{y}), \overline{y}) \notin X_Q(U)$. We then get that z is well-defined because function p' is continuous in its first argument for all $y \in [0, \hat{z}(\overline{y}))$.

The definition of z is such that $X_z \subseteq X_{\hat{z}}$, which implies $X_z \subseteq X_Q(U)$.

Function z is continuous because function $\hat{z}$ is continuous[13] and function p' is continuous on $X_Q(U)$.

Consider first the special case for which $X_z = \emptyset$. This case is such that $p'(y, \overline{y}) = 0$ for all $(y, \overline{y}) \in X$. Hence, we have $z(\overline{y}) = 0$ for all $\overline{y} \geq z_a$. All the necessary properties are trivially satisfied when $X_z = \emptyset$.

There remains the case for which $X_z \neq \emptyset$. We show properties (i) to (v) in turn. Property (iii): function p' is continuous on X_z because Step 2 shows that p' is continuous on $X_Q(U)$ and $X_z \subseteq X_Q(U)$. Property (iv): function p' is weakly decreasing in its first argument on X_z because Step 2 shows that p' has this property on $X_Q(U)$ and $X_z \subseteq X_Q(U)$. Properties (i) and (ii): the definition

[13] Function $\hat{z}$ is continuous in $\overline{y}$ because $\hat{z}(\overline{y}) := \sup\{k \geq 0 | u(z_a, \overline{y}^z) = u(k, \overline{y})$ for some $u \in U\}$ and all functions $u \in U$ are continuous.

of z implies that $p'(y,\overline{y}) = 0$ when $y \geq z(\overline{y})$ and $p'(y,\overline{y}) > 0$ when $y < z(\overline{y})$. This implies that $p'(y,\overline{y}) = 0$ for all $(y,\overline{y}) \in X \backslash X_z$ and $p'(y,\overline{y}) > 0$ for all $(y,\overline{y}) \in X_z$.

There remains to show property (v), i.e., that p' is weakly increasing in its second argument on X_z. Assume to the contrary that there are two bundles $(y,\overline{y}),(y,\overline{y}') \in X_z$ with $\overline{y} < \overline{y}'$ such that $p'(y,\overline{y}) > p'(y,\overline{y}')$. Consider the two distributions $\mathbf{y},\mathbf{y}' \in Y^3$ defined as $\mathbf{y} := (y,\overline{y},\overline{y})$ and $\mathbf{y}' := (y,\overline{y}',\overline{y}')$. By construction, we have $\overline{\mathbf{y}} = \overline{y}$ and $\overline{\mathbf{y}}' = \overline{y}'$. By Lemma 2, we have that $(y_2,\overline{y}),(y_2',\overline{y}'),(y_3,\overline{y}),(y_3',\overline{y}') \notin X_Q(U)$. We also have $u(y_1,\overline{y}) \geq u(y_1',\overline{y}')$ for all $u \in U$ because utility functions are weakly decreasing in the median income. Thus, *Domination* implies that $P_U(\mathbf{y},\mathbf{u}) \leq P_U(\mathbf{y}',\mathbf{u})$ for all $\mathbf{u}$. We also have $p'(y_i,\overline{y}) = p'(y_i',\overline{y}') = 0$ for all $i \in \{2,3\}$ because $X_z \subseteq X_Q(U)$. Therefore, Eq. (S.7) implies that $p'(y_1,\overline{y}) \leq p'(y_1',\overline{y}')$ because $P_U(\mathbf{y},\mathbf{u}) \leq P_U(\mathbf{y}',\mathbf{u})$. This yields a contradiction to $p'(y,\overline{y}) > p'(y,\overline{y}')$. We have thus shown that properties (i) to (v) are all satisfied.

S7 Proof of Proposition 2

$\Rightarrow$. We show that any $P_{\{u\}}$ satisfying the two axioms has the required properties.

By Proposition 1, $P_{\{u\}}$ satisfies *Domination* only if $P_{\{u\}}$ is a fair additive index. As $P_{\{u\}}$ satisfies *Pareto*, $P_{\{u\}}$ also satisfies *Weak Pareto*.[14] By Lemma 4, we have $X_z = X_Q(\{u\}) = X_Q(u)$.

There remains to show that $u(y',\overline{y}') \geq u(y,\overline{y}) \Leftrightarrow p(y',\overline{y}') \leq p(y,\overline{y})$ for all $(y,\overline{y}),(y',\overline{y}') \in X_z$. As u represents a complete ordering on X_z, it is sufficient to show that for all $(y,\overline{y}),(y',\overline{y}') \in X_z$ we have $u(y',\overline{y}') = u(y,\overline{y}) \Rightarrow p(y',\overline{y}') = p(y,\overline{y})$ and $u(y',\overline{y}') > u(y,\overline{y}) \Rightarrow p(y',\overline{y}') < p(y,\overline{y})$.

We start by showing that for any two $(y,\overline{y}),(y',\overline{y}') \in X_z$ we have $u(y',\overline{y}') = u(y,\overline{y}) \Rightarrow p(y',\overline{y}') = p(y,\overline{y})$. Consider the two distributions $\mathbf{y} := (y,\overline{y},\overline{y})$ and $\mathbf{y}' := (y',\overline{y}',\overline{y}')$, for which $\overline{\mathbf{y}} = \overline{y}$ and $\overline{\mathbf{y}}' = \overline{y}'$. We have $(\overline{y},\overline{y}),(\overline{y}',\overline{y}') \notin X_Q(\{u\})$ (Lemma 2). Consider profile $\mathbf{u} \in \{u\}^3$. As $(\overline{y},\overline{y}),(\overline{y}',\overline{y}') \notin X_Q(\{u\})$ and $u(y',\overline{y}') = u(y,\overline{y})$, *Domination* implies $P_{\{u\}}(\mathbf{y},\mathbf{u}) \geq P_{\{u\}}(\mathbf{y}',\mathbf{u})$ and $P_{\{u\}}(\mathbf{y},\mathbf{u}) \leq P_{\{u\}}(\mathbf{y}',\mathbf{u})$, showing that $P_{\{u\}}(\mathbf{y},\mathbf{u}) = P_{\{u\}}(\mathbf{y}',\mathbf{u})$. We have $(\overline{y},\overline{y}),(\overline{y}',\overline{y}') \notin X_z$ because $(\overline{y},\overline{y}),(\overline{y}',\overline{y}') \notin X_Q(\{u\})$ and $X_z = X_Q(u)$. Therefore, we have $p(\overline{y},\overline{y}) = p(\overline{y}',\overline{y}') = 0$ because $P_{\{u\}}$

[14] *Weak Pareto* is a weakening of *Pareto* that we define in Section 4.3. Any fair additive index P_U that satisfies *Pareto* also satisfies *Weak Pareto*. We prove in Lemma 4 that a fair additive index P_U that satisfies *Weak Pareto* has $X_z = X_Q(U)$. We allow ourselves to already reference Lemma 4 in order to avoid duplicating the argument.

is a fair additive index and $(\overline{y}, \overline{y}), (\overline{y}', \overline{y}') \notin X_z$. Therefore, $P_{\{u\}}(\mathbf{y}, \mathbf{u}) = P_{\{u\}}(\mathbf{y}', \mathbf{u})$ implies $p(y, \overline{y}) = p(y', \overline{y}')$ because $P_{\{u\}}$ is a fair additive index, the desired result.

We then show that for any two bundles $(y, \overline{y}), (y', \overline{y}') \in X_z$ we have $u(y', \overline{y}') > u(y, \overline{y}) \Rightarrow p(y', \overline{y}') < p(y, \overline{y})$. Take any bundle $(y'', \overline{y}'') \in X_z$ such that $\overline{y}'' > \overline{y}$ and $u(y'', \overline{y}'') = u(y', \overline{y}')$. By transitivity, we also have $u(y'', \overline{y}'') > u(y, \overline{y})$. As $P_{\{u\}}$ satisfies *Weak Pareto*, Lemma 3 implies that $p(y'', \overline{y}'') < p(y, \overline{y})$. Since $u(y'', \overline{y}'') = u(y', \overline{y}')$, we must have $p(y'', \overline{y}'') = p(y', \overline{y}')$, which shows that $p(y', \overline{y}') < p(y, \overline{y})$, the desired result.

$\Leftarrow$. We show that $P_{\{u\}}$ satisfies the two axioms.

Domination: Take any $(\mathbf{y}, \mathbf{u}), (\mathbf{y}', \mathbf{u}') \in \mathcal{X}_{\{u\}}$ that satisfy the preconditions under which *Domination* implies $P_U(\mathbf{y}', \mathbf{u}') \leq P_U(\mathbf{y}, \mathbf{u})$. That is, we have $n(\mathbf{y}) = n(\mathbf{y}')$ and $u(y'_i, \overline{y}') \geq u(y_i, \overline{y})$ for all $i \in N(\mathbf{y}')$ for whom $(y'_i, \overline{y}') \in X_Q(u)$.[15] In order to prove $P_{\{u\}}(\mathbf{y}', \mathbf{u}') \leq P_{\{u\}}(\mathbf{y}, \mathbf{u})$, we show that $p(y'_i, \overline{y}') \leq p(y_i, \overline{y})$ for all $i \in N(\mathbf{y}')$.

First, consider any $i \in N(\mathbf{y}')$ for whom $(y'_i, \overline{y}') \notin X_Q(u)$. Since $X_z = X_Q(u)$ we have that $(y'_i, \overline{y}') \notin X_z$. Then, the definition of a fair additive index implies that $p(y'_i, \overline{y}') = 0$. This implies $p(y'_i, \overline{y}') \leq p(y_i, \overline{y})$ since $p(y_i, \overline{y}) \geq 0$ by the definition of a fair additive index.

Second, consider any $i \in N(\mathbf{y}')$ for whom $(y'_i, \overline{y}') \in X_Q(u)$. The preconditions of *Domination* imply that $u(y'_i, \overline{y}') \geq u(y_i, \overline{y})$. In turn, this shows that $(y_i, \overline{y}) \in X_Q(u)$. Therefore, we have that $(y_i, \overline{y}), (y'_i, \overline{y}') \in X_z$ because $X_z = X_Q(u)$. This yields the result as $u(y'_i, \overline{y}') \geq u(y_i, \overline{y}) \Rightarrow p(y'_i, \overline{y}') \leq p(y_i, \overline{y})$ when $(y_i, \overline{y}), (y'_i, \overline{y}') \in X_z$.

Pareto: Take any $(\mathbf{y}, \mathbf{u}), (\mathbf{y}', \mathbf{u}) \in \mathcal{X}_{\{u\}}$ that satisfy the preconditions under which *Pareto* implies $P_U(\mathbf{y}', \mathbf{u}) \leq P_U(\mathbf{y}, \mathbf{u})$. That is, we have $n(\mathbf{y}) = n(\mathbf{y}')$, $u_i(y'_i, \overline{y}') \geq u_i(y_i, \overline{y})$ for all $i \in N(\mathbf{y}')$. The unanimous preference for distribution $\mathbf{y}'$ implies that $\overline{y} \leq \overline{y}'$ (Lemma 1). In order to prove $P_{\{u\}}(\mathbf{y}', \mathbf{u}') \leq P_{\{u\}}(\mathbf{y}, \mathbf{u})$, we show $p(y'_i, \overline{y}') \leq p(y_i, \overline{y})$ for all $i \in N(\mathbf{y}')$.

Consider any $i \in N(\mathbf{y}')$. The preconditions of *Pareto* imply that $u(y'_i, \overline{y}') \geq u(y_i, \overline{y})$. The argument is the same as that given in the proof of *Domination*.

Take any $(\mathbf{y}, \mathbf{u}), (\mathbf{y}', \mathbf{u}) \in \mathcal{X}_{\{u\}}$ that satisfy the preconditions under which *Pareto* implies $P_U(\mathbf{y}', \mathbf{u}) < P_U(\mathbf{y}, \mathbf{u})$. Following the same argument, we have that $p(y'_i, \overline{y}') \leq p(y_i, \overline{y})$ for all $i \in N(\mathbf{y}) = N(\mathbf{y}')$. In addition, we have $u_\ell(y'_\ell, \overline{y}') >$

[15] On $\{u\}$, we have $\mathbf{u} = \mathbf{u}' = (u, \ldots, u)$.

$u_\ell(y_\ell, \overline{y})$ for some $\ell \in Q(\mathbf{y}, \mathbf{u})$. We must show that $p(y'_\ell, \overline{y}') < p(y_\ell, \overline{y})$. As $\ell \in Q(\mathbf{y}, \mathbf{u})$, we have $(y_\ell, \overline{y}) \in X_Q(u)$. Therefore we have $p(y_\ell, \overline{y}) > 0$ because $(y_\ell, \overline{y}) \in X_z$ since $X_z = X_Q(u)$. If $(y'_\ell, \overline{y}') \notin X_z$, then $p(y'_\ell, \overline{y}') = 0$, which yields the result. Otherwise $(y'_\ell, \overline{y}') \in X_z$, and we then have $u(y'_\ell, \overline{y}') > u(y_\ell, \overline{y}) \Rightarrow p(y'_\ell, \overline{y}') < p(y_i, \overline{y})$ because $(y_\ell, \overline{y}), (y'_\ell, \overline{y}') \in X_z$.

S8 Relation between *Weak Pareto* and the *Weak Relativity* Axiom

Ravallion and Chen (2011) use another welfare-consistency requirement. These authors note that the poverty index must be reduced when all incomes grow in the same proportion. This requirement can be expressed in our framework as follows.

Axiom S.2 (*Weak Relativity*)**.**
For all $(\mathbf{y}, \mathbf{u}) \in \mathcal{X}_U$ *and* $\lambda > 1$, *if* $\mathbf{y}' = \lambda \mathbf{y}$ *and* $y_\ell > 0$ *for some* $\ell \in Q(\mathbf{y}, \mathbf{u})$, *then* $P_U(\mathbf{y}', \mathbf{u}) < P_U(\mathbf{y}, \mathbf{u})$.

Proposition S.4 shows that *Weak Relativity* is a weakening of *Weak Pareto*.

Proposition S.4.
Given any $U \subseteq U^B$, *the additive index* P_U *satisfies Weak Pareto only if* P_U *satisfies Weak Relativity.*

Proof. We show that the preconditions of *Weak Pareto* for a strict comparison are met when the preconditions of *Weak Relativity* are met.

Take any $(\mathbf{y}, \mathbf{u}), (\mathbf{y}', \mathbf{u}) \in \mathcal{X}_{U^B}$ that satisfy the preconditions under which *Weak Relativity* implies $P_U(\mathbf{y}', \mathbf{u}) < P_U(\mathbf{y}, \mathbf{u})$. That is, $\mathbf{y}' = \lambda \mathbf{y}$ for some $\lambda > 1$ and $y_\ell > 0$ for some $\ell \in Q(\mathbf{y}, \mathbf{u})$.

First, we show that $y'_j \geq \frac{\overline{y}'}{\overline{y}} y_j$ for all $j \notin Q(\mathbf{y}, \mathbf{u})$. As $\mathbf{y}' = \lambda \mathbf{y}$, we have $y'_i = \frac{\overline{y}'}{\overline{y}} y_i$ for all $i \in N(\mathbf{y})$.

Second, we show that $u_i(y'_i, \overline{y}') \geq u_i(y_i, \overline{y})$ for all $i \in N(\mathbf{y})$. For all $i \in N(\mathbf{y})$ for whom $y_i > 0$ we have $u_i(y'_i, \overline{y}') > u_i(y_i, \overline{y})$ because $y'_i = \frac{\overline{y}'}{\overline{y}} y_i$ and utility functions are strictly increasing in own income when relative income is held constant. For all $i \in N(\mathbf{y})$ for whom $y_i = 0$ we have $u_i(y'_i, \overline{y}') = u_i(y_i, \overline{y})$ because we have $y'_i = y_i = 0$, which implies that $u_i(y'_i, y'_i/\overline{y}') = u_i(y_i, y_i/\overline{y}) = u_i(0, 0)$.

Finally, we show that $u_\ell(y_\ell, \overline{y}') > u_\ell(y_\ell, \overline{y})$ for some $\ell \in Q(\mathbf{y}, \mathbf{u})$. This follows from the fact that there is $y_\ell > 0$ for some $\ell \in Q(\mathbf{y}, \mathbf{u})$, for whom $y'_\ell = \frac{\overline{y}'}{\overline{y}} y_\ell > y_\ell$. ∎

S9 Proof of Theorem 2

The proof is based on Lemmas S.2 and S.3.

Lemma S.2.
Given any $U \subseteq U^B$, for all $\overline{y} \geq \overline{y}^z$ we have $(\frac{z_a}{\overline{y}^z}\overline{y}, \overline{y}) \notin X_Q(U)$.

Proof. Let $\lambda := \overline{y}/\overline{y}^z$, which is such that $\lambda \geq 1$. Since utility is increasing in own income when holding relative income constant, we have $u(\lambda z_a, \lambda \overline{y}^z) \geq u(z_a, \overline{y}^z)$ for all $u \in U$. This shows that $u(\frac{\overline{y}}{\overline{y}^z} z_a, \frac{\overline{y}}{\overline{y}^z}\overline{y}^z) \geq u(z_a, \overline{y}^z)$ for all $u \in U$ and thus $(\frac{z_a}{\overline{y}^z}\overline{y}, \overline{y}) \notin X_Q(U)$. ■

Lemma S.3.
For all $(y, \overline{y}) \in X_Q(U^B) \backslash X_A$, there exists a utility function $u \in U^B$ such that $(y, \overline{y}) \in X_Q(u)$ and $u(y, \overline{y}') = u(y, \overline{y})$ for all $\overline{y}' \geq \overline{y}$.

Proof. The proof is by construction. In Step 1, we construct a particular indifference curve that passes through bundle $(y, \overline{y})$, below the reference bundle $(z_a, \overline{y}^z)$ and that is flat for all $\overline{y}' \geq \overline{y}$. In Step 2, we construct a utility function $u \in U^B$ that has one of its indifference curve that corresponds to the indifference curve constructed in the first step. If it is the case, we have indeed that $(y, \overline{y}) \in X_Q(u)$ and $u(y, \overline{y}') = u(y, \overline{y})$ for all $\overline{y}' \geq \overline{y}$.

Step 1. The construction of the indifference curve is illustrated in Figure S.4. Take any $s \in \left(\frac{y - z_a}{\overline{y} - \overline{y}^z}, \frac{y}{\overline{y}}\right)$ and let $R := y - s\overline{y}$.[16] Observe that we have $s > 0$ because $y \geq z_a$ as $(y, \overline{y}) \notin X_A$. We also have $R > 0$ because $s < y/\overline{y}$. The indifference curve is defined by the function $w' : \mathbb{R}_+ \to \mathbb{R}_{++}$,

$$w'(\overline{y}') := \begin{cases} R + s\overline{y}' & \text{if } \overline{y}' < \overline{y}, \\ \\ R + s\overline{y} & \text{if } \overline{y}' \geq \overline{y}. \end{cases} \tag{S.8}$$

This indifference curve passes through $(y, \overline{y})$ since $w'(\overline{y}) = y$. This indifference curve passes below the reference bundle because we have $w'(\overline{y}^z) < z_a$ since $s > \frac{y - z_a}{\overline{y} - \overline{y}^z}$. This indifference curve is flat beyond $(y, \overline{y})$ because w' is constant for all $\overline{y}' \geq \overline{y}$.

Step 2. We construct a utility function $u \in U^B$ such that $u(w'(\overline{y}'), \overline{y}') = u(w'(\overline{y}''), \overline{y}'')$ for all $\overline{y}', \overline{y}'' \in [z_a, \infty)$. The utility function u is defined for all

[16] We have that $\frac{y - z_a}{\overline{y} - \overline{y}^z} < \frac{y}{\overline{y}}$ because we have $y < \frac{z_a}{\overline{y}^z}\overline{y}$, which itself follows from the fact that $(y, \overline{y}) \in X_Q(U^B) \backslash X_A$ and $X_{z^{**}} = X_Q(U^B)$ and $z^{**}(\overline{y}) = \frac{z_a}{\overline{y}^z}\overline{y}$.

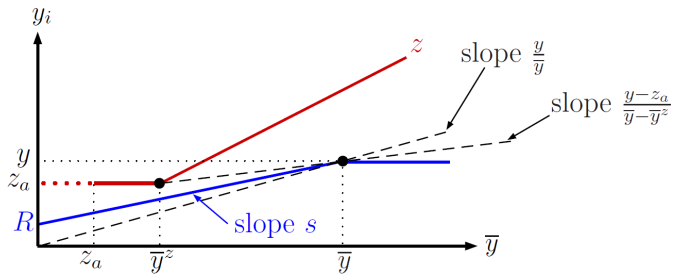


Figure S.4: Construction of the indifference curve w' (in blue) passing through bundle $(y, \overline{y})$.

$(y', \overline{y}') \in X$ as

$$u(y', \overline{y}') := \frac{y'}{w'(\overline{y}')}. \tag{S.9}$$

The construction is such that $u(w'(\overline{y}'), \overline{y}') = u(w'(\overline{y}''), \overline{y}'') = 1$ for all $\overline{y}', \overline{y}'' \in [z_a, \infty)$. There remains to show that $u \in U^B$. First, u is continuous because w' is continuous. Then, u is strictly increasing in own income. Also, u is weakly decreasing in the median income because w' is a weakly increasing function. Finally, we must show that u is strictly increasing in own income when holding relative income constant. For all $\overline{y}' \geq \overline{y}$, this follows from the fact that u is strictly increasing in own income and independent on the median income, i.e., independent on relative income. For all $\overline{y}' < \overline{y}$, we can rewrite u for any $y' \geq 0$ as $u(y', \overline{y}') = \frac{1}{\frac{R}{y'} + \frac{s}{y'/\overline{y}'}}$, which shows that u has the required property. ■

$\Rightarrow$. We show that any P_{U^B} satisfying these two axioms has the required properties.

As the self-centered preference $u^0 \in U^B$, we have $U^* \cap U^B \neq \emptyset$. By Proposition 4, if P_{U^B} satisfies *Domination* and *Weak Pareto*, then P_{U^B} is a hierarchical index. By definition of a hierarchical index, we have $X_z = X_Q(U^B)$.

We show that $X_z = X_{z^{**}}$, i.e. $z(\overline{y}) = z^{**}(\overline{y})$ for all $\overline{y} \geq z_a$. The proof exploits the fact that $X_z = X_Q(U^B)$.

The argument showing that $z(\overline{y}) = z_a$ for all $\overline{y} \leq \overline{y}^z$ is the same as that given in the proof of Theorem 1.

We then show that $z(\overline{y}) = \frac{z_a}{\overline{y}^z}\overline{y}$ for all $\overline{y} > \overline{y}^z$. Take any $\overline{y} > \overline{y}^z$. We start by showing that $z(\overline{y}) \leq \frac{z_a}{\overline{y}^z}\overline{y}$. As $X_z = X_Q(U^B)$, we must have $z(\overline{y}) \leq y'$ for all y' such that $(y', \overline{y}) \notin X_Q(U^B)$. By Lemma S.2, we have $(\frac{z_a}{\overline{y}^z}\overline{y}, \overline{y}) \notin X_Q(U^B)$, which implies $z(\overline{y}) \leq \frac{z_a}{\overline{y}^z}\overline{y}$. There remains to show that $z(\overline{y}) \geq \frac{z_a}{\overline{y}^z}\overline{y}$. Take any

$y \in [z_a, \frac{z_a}{\overline{y}^z}\overline{y})$, it is sufficient to show that there exists some $u^\sigma \in U^B$ such that $(y, \overline{y}) \in X_Q(u^\sigma)$. Indeed, we would then have $(y, \overline{y}) \in X_Q(U^B)$, which directly implies $z(\overline{y}) > y$ as $X_z = X_Q(U^B)$. This in turn yields $z(\overline{y}) \geq \frac{z_a}{\overline{y}^z}\overline{y}$ because $z(\overline{y}) > y$ for all $y \in [z_a, \frac{z_a}{\overline{y}^z}\overline{y})$. There remains to prove that such $u^\sigma \in U^B$ exists. There exists some $y' \in (y, \frac{z_a}{\overline{y}^z}\overline{y})$ because $y \in [z_a, \frac{z_a}{\overline{y}^z}\overline{y})$. By Lemma 5, Parts (i) and (ii), we have $u^{\sigma^*}(y', \overline{y}) = u^{\sigma^*}(z_a, \overline{y}^z)$ for some $\sigma^* \geq 0$.[17] This implies that $u^{\sigma^*}(y, \overline{y}) < u^{\sigma^*}(z_a, \overline{y}^z)$ because $y' > y$. The preference u^{σ^*} has the required properties, as desired. We have thus shown that $z(\overline{y}) = z^{**}(\overline{y})$ for all $\overline{y} \geq z_a$.

There remains to show that the poverty score function p has the required properties. Assume to the contrary that there are two bundles $(y, \overline{y}), (y, \overline{y}') \in X_z \backslash X_A$ with $\overline{y} < \overline{y}'$ such that $p(y, \overline{y}) \neq p(y, \overline{y}')$. As $X_z = X_Q(U^B)$, we thus have $(y, \overline{y}), (y, \overline{y}') \in X_Q(U^B) \backslash X_A$. As function p is weakly increasing in the median income, this implies that $p(y, \overline{y}) < p(y, \overline{y}')$. By Lemma S.3, there exists a preference $u' \in U^B$ such that $(y, \overline{y}) \in X_Q(u')$ and $u'(y, \overline{y}') = u'(y, \overline{y})$. As $\overline{y} < \overline{y}'$ and P_{U^B} is a fair additive index satisfying *Weak Pareto*, Lemma 3 implies that $p(y, \overline{y}) \geq p(y, \overline{y}')$, the desired contradiction.

$\Leftarrow$. We show that P_{U^B} satisfies the two axioms.

Domination: Take any $(\mathbf{y}, \mathbf{u}), (\mathbf{y}', \mathbf{u}') \in \mathcal{X}_{U^B}$ that satisfy the preconditions under which *Domination* implies $P_{U^B}(\mathbf{y}', \mathbf{u}') \leq P_{U^B}(\mathbf{y}, \mathbf{u})$. That is, we have $n(\mathbf{y}) = n(\mathbf{y}')$ and $u(y_i', \overline{y}') \geq u(y_i, \overline{y})$ for all $i \in N(\mathbf{y}')$ and all $u \in U^B$ such that $(y_i', \overline{y}') \in X_Q(u)$. In order to prove $P_{U^B}(\mathbf{y}', \mathbf{u}') \leq P_{U^B}(\mathbf{y}, \mathbf{u})$, we show that $p(y_i', \overline{y}') \leq p(y_i, \overline{y})$ for all $i \in N(\mathbf{y}')$.

Take any $i \in N(\mathbf{y}')$ for whom $(y_i', \overline{y}') \notin X_z$. Since P_{U^B} is a fair additive index, we have $p(y_i', \overline{y}') = 0$ and thus $p(y_i', \overline{y}') \leq p(y_i, \overline{y})$.

Take any $i \in N(\mathbf{y}')$ for whom $(y_i', \overline{y}') \in X_z$. We show that $y_i' \geq y_i$ and $(y_i, \overline{y}), (y_i', \overline{y}') \in X_z$, which implies $p(y_i', \overline{y}') \leq p(y_i, \overline{y})$ by the properties of function p. First, we show that $(y_i, \overline{y}) \in X_z$. As $X_z = X_Q(U^B)$, there exists some $u' \in U^B$ such that $(y_i', \overline{y}') \in X_Q(u')$. Therefore, the precondition of *Domination* implies that $u'(y_i', \overline{y}') \geq u'(y_i, \overline{y})$. In turn, we have $(y_i, \overline{y}) \in X_Q(u')$ because $(y_i', \overline{y}') \in X_Q(u')$ and $u'(y_i', \overline{y}') \geq u'(y_i, \overline{y})$. This shows that $(y_i, \overline{y}) \in X_z$ because $X_z = X_Q(U^B)$. There remains to show that $y_i' \geq y_i$. Assume to the contrary that

[17] We can invoke Lemma 5 if $y' < z^*$ where $z^* := R + R\bar{\sigma}\overline{y}$ and $R := \frac{z_a}{1+\bar{\sigma}\overline{y}^z}$. By definition, $z^* \to \frac{z_a}{\overline{y}^z}\overline{y}$ as $\bar{\sigma} \to \infty$. Hence, there must exist some large enough $\bar{\sigma} > 0$ for which $y' < z^*$ because $y' < \frac{z_a}{\overline{y}^z}\overline{y}$.

$y_i > y'_i$. We have $(y'_i, \overline{y}) \in X_z$ because $(y_i, \overline{y}) \in X_z$ and $y_i > y'_i$. By Lemma S.3, there exists some $u \in U^B$ such that $u(y'_i, \overline{y}') = u(y'_i, \overline{y})$ and $(y'_i, \overline{y}') \in X_Q(u)$. Since $y_i > y'_i$, we must have $u(y'_i, \overline{y}') < u(y_i, \overline{y})$. This is a contradiction to the precondition of *Domination* that requires $u(y'_i, \overline{y}') \geq u(y_i, \overline{y})$ because $(y'_i, \overline{y}') \in X_Q(u)$, the desired result.

Weak Pareto: Take any $(\mathbf{y}, \mathbf{u}), (\mathbf{y}', \mathbf{u}) \in \mathcal{X}_{U^B}$ that satisfy the preconditions under which *Weak Pareto* implies $P_{U^B}(\mathbf{y}', \mathbf{u}) \leq P_{U^B}(\mathbf{y}, \mathbf{u})$. That is, we have $n(\mathbf{y}) = n(\mathbf{y}')$, $u_i(y'_i, \overline{y}') \geq u_i(y_i, \overline{y})$ for all $i \in N(\mathbf{y}')$ and $y'_j \geq \frac{\overline{y}'}{\overline{y}} y_j$ for all $j \notin Q(\mathbf{y}, \mathbf{u})$. The unanimous preference for distribution $\mathbf{y}'$ implies that $\overline{y} \leq \overline{y}'$ (Lemma 1). In order to prove $P_{U^B}(\mathbf{y}', \mathbf{u}) \leq P_{U^B}(\mathbf{y}, \mathbf{u})$, we show that $p(y'_i, \overline{y}') \leq p(y_i, \overline{y})$ for all $i \in N(\mathbf{y}')$.

Take any $i \in N(\mathbf{y}')$ for whom $(y'_i, \overline{y}') \notin X_z$. Since P_{U^B} is a fair additive index, we have $p(y'_i, \overline{y}') = 0$ and thus $p(y'_i, \overline{y}') \leq p(y_i, \overline{y})$.

Finally, take any $i \in N(\mathbf{y}')$ for whom $(y'_i, \overline{y}') \in X_z$.

First, we show this case is such that $(y_i, \overline{y}) \in X_z$. Indeed, if $(y_i, \overline{y}) \notin X_z$, then $(y_i, \overline{y}) \notin X_Q(U^B)$. Therefore, $i \notin Q(\mathbf{y}, \mathbf{u})$ and the precondition of *Weak Pareto* requires $y'_i \geq \frac{\overline{y}'}{\overline{y}} y_i$. This implies in turn that $u(y'_i, \overline{y}') \geq u(y_i, \overline{y})$ for all $u \in U^B$ because $\overline{y}' \geq \overline{y}$ and utility is increasing in own income when relative income is kept constant. As $(y_i, \overline{y}) \notin X_Q(U^B)$ and $u(y'_i, \overline{y}') \geq u(y_i, \overline{y})$ for all $u \in U^B$, we must have that $(y'_i, \overline{y}') \notin X_Q(U^B)$. We therefore get a contradiction to $(y'_i, \overline{y}') \in X_z$ because $X_z = X_Q(U^B)$

Now, the other precondition of *Weak Pareto* requires that $u_i(y'_i, \overline{y}') \geq u_i(y_i, \overline{y})$ for all $i \in N(\mathbf{y}')$. The two inequalities $\overline{y} \leq \overline{y}'$ and $u_i(y'_i, \overline{y}') \geq u_i(y_i, \overline{y})$ together imply that $y'_i \geq y_i$ because individual utility functions are weakly decreasing in the median income. As $(y_i, \overline{y}), (y'_i, \overline{y}') \in X_z$ and $y'_i \geq y_i$, the properties of function p yield $p(y'_i, \overline{y}') \leq p(y_i, \overline{y})$, the desired result.

Take any $(\mathbf{y}, \mathbf{u}), (\mathbf{y}', \mathbf{u}) \in \mathcal{X}_{U^B}$ that satisfy the preconditions under which *Weak Pareto* implies $P_{U^B}(\mathbf{y}', \mathbf{u}) < P_{U^B}(\mathbf{y}, \mathbf{u})$. The proof is a straightforward adaptation of arguments used above, and is thus omitted.

S10 Robustness of main theorem on U^C

Theorem S.1.

The additive index P_{U^C} satisfies Domination and Weak Pareto if and only if P_{U^C}

is a hierarchical index with global line z^{**} defined for all $\overline{y} \geq z_a$ as

$$z^{**}(\overline{y}) := \max\left(z_a, \frac{z_a}{\overline{y}^z}\overline{y}\right),$$

where $R := \frac{z_a}{1+\bar{\sigma}\overline{y}^z}$, and whose poverty score function p is such that for all $(y, \overline{y}), (y', \overline{y}') \in X_{z^{**}} \backslash X_A$[18]

$$p(y, \overline{y}) = p(y', \overline{y}') \qquad \textit{when} \qquad \frac{y - z_a}{\overline{y} - \overline{y}^z} = \frac{y' - z_a}{\overline{y}' - \overline{y}^z}.$$

Proof. $\Rightarrow$. We show that any P_{U^C} satisfying these two axioms has the required properties.

As the self-centered preference $u^0 \in U^C$, we have $U^* \cap U^C \neq \emptyset$. By Proposition 4, if P_{U^C} satisfies *Domination* and *Weak Pareto*, then P_{U^C} is a hierarchical index. By definition of a hierarchical index, we have $X_z = X_Q(U^C)$.

The proof that $X_z = X_{z^{**}}$, i.e. $z(\overline{y}) = z^{**}(\overline{y})$ for all $\overline{y} \geq z_a$, follows the same argument as that presented for Theorem 1 (for the case $\bar{\sigma} \to \infty$) and is thus omitted.

There remains to show that for all $(y, \overline{y}), (y', \overline{y}') \in X_{z^{**}} \backslash X_A$ with $\frac{y-z_a}{\overline{y}-\overline{y}^z} = \frac{y'-z_a}{\overline{y}'-\overline{y}^z}$ we have $p(y, \overline{y}) = p(y', \overline{y}')$. Assume to the contrary that there are two bundles $(y, \overline{y}), (y', \overline{y}') \in X_{z^{**}} \backslash X_A$ with $\frac{y-z_a}{\overline{y}-\overline{y}^z} = \frac{y'-z_a}{\overline{y}'-\overline{y}^z}$ but $p(y, \overline{y}) \neq p(y', \overline{y}')$. Without loss of generality, assume that $\overline{y} < \overline{y}'$. Since $(y, \overline{y}) \in X_{z^{**}} \backslash X_A$, we have $\overline{y} > \overline{y}^z$ and $y \geq z_a$. Since $\overline{y} < \overline{y}'$, we have $y \leq y'$.

- Case 1: $p(y, \overline{y}) < p(y', \overline{y}')$.

  Since $U^{\bar{\sigma}} \subset U^C$, the argument yielding a contradiction is the same as that given in the proof of Theorem 1.

- Case 2: $p(y, \overline{y}) > p(y', \overline{y}')$.

  Consider the two distributions $\mathbf{y} := (y, \overline{y}, \overline{y})$ and $\mathbf{y}' := (y', \overline{y}', \overline{y}')$, which are, respectively, such that $\overline{\mathbf{y}} = \overline{y}$ and $\overline{\mathbf{y}}' = \overline{y}'$. We have that bundles $(\overline{y}, \overline{y}), (\overline{y}', \overline{y}') \notin X_Q(U^C)$ (Lemma 2).

  There exists some $u \in U^C$ for which $(y, \overline{y}) \in X_Q(u)$ because $(y, \overline{y}) \in X_{z^{**}}$ and $X_{z^{**}} = X_Q(U^C)$ (Lemma 4). Let $\mathbf{u} = (u, u, u) \in (U^C)^3$.

[18] Recall that, by definition, a hierarchical index also has for all $(y, \overline{y}), (y', \overline{y}') \in X_A$ that $p(y, \overline{y}) = p(y', \overline{y}')$ when $y = y'$.

As $X_{z^{**}} = X_Q(U^C)$, we have $p(y_i, \overline{y}) = p(y'_i, \overline{y}') = 0$ for all $i \in \{2, 3\}$. Since $p(y_1, \overline{y}) > p(y'_1, \overline{y}')$, we must have $P_{U^C}(\mathbf{y}, \mathbf{u}) > P_{U^C}(\mathbf{y}', \mathbf{u})$.

We derive a contradiction by showing that *Domination* implies $P_{U^C}(\mathbf{y}, \mathbf{u}) \leq P_{U^C}(\mathbf{y}', \mathbf{u})$. By construction, we have $Q(\mathbf{y}', \mathbf{u}') \subseteq \{1\}$ for all $\mathbf{u}' \in (U^C)^3$. Therefore, *Domination* implies $P_{U^C}(\mathbf{y}, \mathbf{u}) \leq P_{U^C}(\mathbf{y}', \mathbf{u})$ if for all $\mathbf{u}' \in (U^C)^3$ for which $Q(\mathbf{y}', \mathbf{u}') = \{1\}$ (i.e. such that $(y', \overline{y}') \in X_Q(u'_1)$) we have $u'_1(y_1, \overline{y}) > u'_1(y'_1, \overline{y}')$. Letting $\mathbf{u}''$ be any profile in $(U^C)^3$ for which $Q(\mathbf{y}', \mathbf{u}'') = \{1\}$, we show that $u''_1(y_1, \overline{y}) > u''_1(y'_1, \overline{y}')$. The constructions are illustrated in Figure S.5. Let the income level y^z be implicitly defined by $u''_1(y^z, \overline{y}^z) = u''_1(y'_1, \overline{y}')$. We must have $y^z < z_a$ because $u''_1(y'_1, \overline{y}') < u''_1(z_a, \overline{y}^z)$ as $Q(\mathbf{y}', \mathbf{u}'') = \{1\}$. The indifference curves associated to u''_1 are convex because $u''_1 \in U^C$. As shown in Figure S.5, the convexity of the indifference curve of u''_1 passing through $(y'_1, \overline{y}')$ and $(y^z, \overline{y}^z)$ implies that

$$u''_1\left(y^z + \frac{y' - y^z}{\overline{y}' - \overline{y}^z}(\overline{y} - \overline{y}^z), \overline{y}\right) \geq u''_1(y'_1, \overline{y}')$$

because $\overline{y}^z < \overline{y} < \overline{y}'$. This implies that $u''_1(y_1, \overline{y}) > u''_1(y'_1, \overline{y}')$ because

$$y > y^z + \frac{y' - y^z}{\overline{y}' - \overline{y}^z}(\overline{y} - \overline{y}^z),$$

where the last inequality follows from $y = z_a + \frac{y' - y^z}{\overline{y}' - \overline{y}^z}(\overline{y} - \overline{y}^z)$ and $y^z < z_a$.

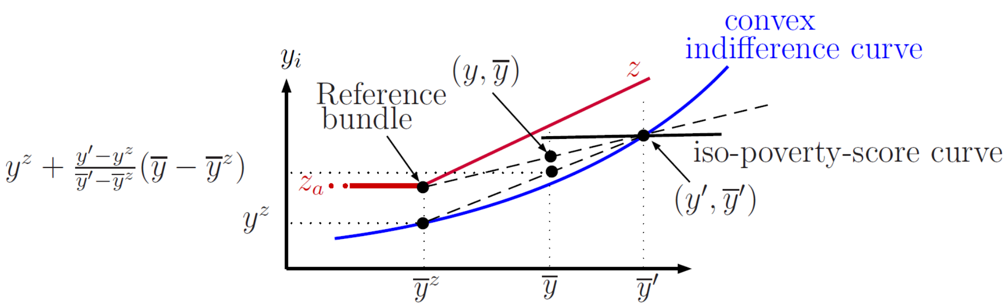


Figure S.5: Implications of convexity of the indifference curve (in blue) passing through $(y^z, \overline{y}^z)$ and $(y', \overline{y}')$.

$\Leftarrow$. We show that P_{U^C} satisfies the two axioms.

Domination: Since $U^{\bar{\sigma}} \subset U^C$, the argument is the same as that given in the proof of Theorem 1.

Weak Pareto: Since $U^{\bar{\sigma}} \subset U^C$, the argument is almost the same as that given in the proof of Theorem 1.

The only difference lies in proving that for any $i \in N(\mathbf{y}')$ for whom $(y'_i, \overline{y}') \in X_{z^{**}} \backslash X_A$, $(y_i, \overline{y}) \in X_{z^{**}} \backslash X_A$ and $i \in Q(\mathbf{y}, \mathbf{u})$, the preconditions of *Weak Pareto* imply that[19]

$$\frac{y'_i - z_a}{\overline{y}' - \overline{y}^z} \geq \frac{y_i - z_a}{\overline{y} - \overline{y}^z}.$$

The preconditions of *Weak Pareto* imply that $\overline{y} \leq \overline{y}'$ (Lemma 1). As $i \in Q(\mathbf{y}, \mathbf{u})$, these preconditions also imply that $u_i(y'_i, \overline{y}') \geq u_i(y_i, \overline{y})$. The reasoning is illustrated in Figure S.6. As $i \in Q(\mathbf{y}, \mathbf{u})$, the indifference curve of u_i passing through $(y_i, \overline{y})$ must pass *below* the reference bundle $(z_a, \overline{y}^z)$. As $u_i(y'_i, \overline{y}') \geq u_i(y_i, \overline{y})$, the indifference curve of u_i passing through $(y_i, \overline{y})$ *cannot pass above* bundle $(y'_i, \overline{y}')$. The indifference curves associated to u_i are convex because $u_i \in U^C$. Together, the indifference curve of u_i passing through $(y_i, \overline{y})$ satisfies these constraints and is convex only if

$$y'_i \geq z_a + \frac{y_i - z_a}{\overline{y} - \overline{y}^z}(\overline{y}' - \overline{y}^z),$$

because $\overline{y} \leq \overline{y}'$. A few manipulations yield $\frac{y'_i - z_a}{\overline{y}' - \overline{y}^z} \geq \frac{y_i - z_a}{\overline{y} - \overline{y}^z}$.

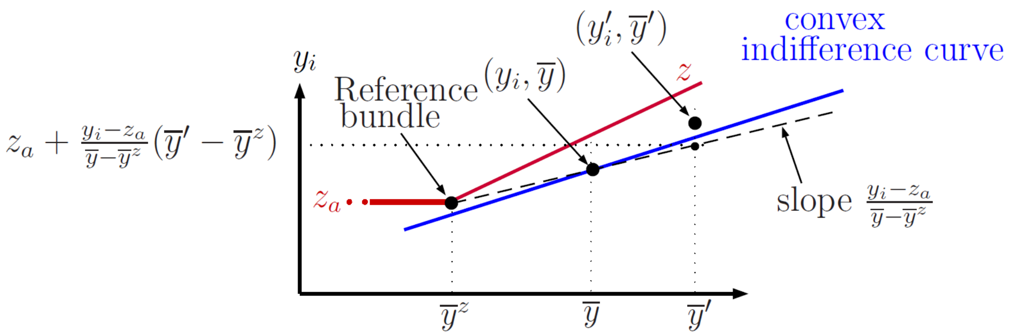


Figure S.6: Implications of the convexity of the indifference curve (in blue) passing through $(y_i, \overline{y})$.

■

[19]More precisely, in the proof of Theorem 1, we can no longer consider $u^{\hat{\sigma}} := u_i$ in order to prove the existence the utility function $u^{\hat{\sigma}}$ with the claimed properties because there is no guarantee that $u_i \in U^{\bar{\sigma}}$.

S11 Utility strictly increasing in relative income

We explain why Theorem 1 is robust to a framework for which utility is *strictly* increasing in relative income. Let $U^{\bar{\sigma}0}$ be the subset of utility functions u^σ defined by Eq. (1) for which $0 < \sigma < \bar{\sigma}$ for some $\bar{\sigma} > 0$. The set $U^{\bar{\sigma}0}$ does not contain self-centered preferences (because $\sigma \neq 0$).

The proof of an equivalent of Theorem 1 for set $U^{\bar{\sigma}0}$ is almost identical, except when we show that $P_{U^{\bar{\sigma}0}}$ is a hierarchical index. We cannot invoke Proposition 4 because $U^* \cap U^{\bar{\sigma}0} = \emptyset$ since $\sigma > 0$. By Proposition 1, $P_{U^{\bar{\sigma}0}}$ satisfies *Domination* only if $P_{U^{\bar{\sigma}0}}$ is a fair additive index. By Lemma 4, any fair additive index that satisfies *Weak Pareto* is such that $X_z = X_Q(U^{\bar{\sigma}0})$. The proof that p is strictly decreasing in its first argument on X_z is the same as in the proof of Proposition 4 and is thus omitted. There remains to show that for all $(y, \overline{y}), (y, \overline{y}') \in X_A \cap X_z$, we have $p(y, \overline{y}) = p(y, \overline{y}')$. Consider the contradiction assumption that for $y \in [0, z_a)$ and $\overline{y} < \overline{y}'$ we have $p(y, \overline{y}) \neq p(y, \overline{y}')$. Since $P_{U^{\bar{\sigma}0}}$ is a fair additive index, p is weakly increasing in its second argument, and thus we must have $p(y, \overline{y}) < p(y, \overline{y}')$. As p is continuous on X_z, we have that p is continuous on X_A because $X_A \subseteq X_z$.[20] As p is continuous on X_A, there exists some $\epsilon \in (0, z_a - y)$ such that $p(y, \overline{y}) < p(y + \epsilon, \overline{y}')$. As $\epsilon > 0$ and $\overline{y} < \overline{y}'$, there exists some sufficiently small $\sigma' > 0$ such that $u^{\sigma'}(y, \overline{y}) < u^{\sigma'}(y + \epsilon, \overline{y}')$. As $(y, \overline{y}) \in X_Q(U^{\bar{\sigma}0})$, Lemma 3 ($P_{U^{\bar{\sigma}0}}$ satisfies *Weak Pareto*) implies $p(y, \overline{y}) > p(y + \epsilon, \overline{y}')$, the desired contradiction.

[20] Here is why $X_A \subseteq X_z$. For any $(y, \overline{y}) \in X_A$ with $\overline{y} \geq \overline{y}^z$, this follows from the fact that utility functions are increasing in relative income when holding own income constant. For any $(y, \overline{y}) \in X_A$ with $\overline{y} < \overline{y}^z$, we have $(y, \overline{y}) \in X_Q(u^\sigma)$ for any $u^\sigma \in U^{\bar{\sigma}0}$ with σ sufficiently small.

References

Atkinson, A. and Bourguignon, F. (2001). Poverty and Inclusion from a World Perspective. In Stiglitz, J. and Muet, P.-A., editors, *Governance, Equity and Global Markets*. Oxford University Press, New York.

Cerioli, A. and Zani, S. (1990). A fuzzy approach to the measurement of poverty. In *Income and wealth distribution, inequality and poverty*, pages 272–284. Springer.

Ravallion, M. (2020). On measuring global poverty. *Annual Review of Economics*, 12(1):167–188.

Ravallion, M. and Chen, S. (2011). Weakly relative poverty. *Review of Economics and Statistics*, 93(4):1251–1261.

Sen, A. (1985). Commodities and capabilities. In *Professor Dr. P. Hennipman Lectures in Economics: Theory, Institutions, Policy*, volume 7. Elsevier, Amsterdam.

Smith, A. (1776). An inquiry into the nature and causes of the wealth of nations: Volume one. London: printed for W. Strahan; and T. Cadell, 1776.

Townsend, P. (1979). *Poverty in the United Kingdom*. Penguin Books, Harmondsworth, Middlesex.

Zheng, B. (2015). Poverty: fuzzy measurement and crisp ordering. *Social Choice and Welfare*, 45(1):203–229.